Logic and Proofs

Proportion:

A proportion (or statement) is a declarative sentence which is either true or false but not both.

Note:

① If a proportion is true its truth value is denoted by 'T'.

② If a proportion is false then its truth value is denoted by 'F'.

③ P, Q, R, ... are used to denote proportions.

Connectives:

① Negation (NOT) (~) ($\overset{not}{\neg}$)

P	¬P
T	F
F	T

② Conjunction (AND or ∧) cap

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

③ Disjunction (OR or ∨) cap

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

④ Conditional (If ... then or $\xrightarrow{\text{implies}}$)

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

⑤ Biconditional (If and only if (or) Iff (or) $\rightleftarrows$ (or) $\leftrightarrow$)

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

1) Write in symbolic form

(i) Malathy is poor but happy.

(ii) Malathy is rich or unhappy.

(iii) Malathy is neither rich nor happy.

Sol :

P : Malathy is poor

Q : Malathy is happy

(i) $P \wedge Q$

(ii) $\neg P \vee \neg Q \quad = \neg(P \wedge Q)$

(iii) $P \wedge \neg Q$

Logical equivalences or Equivalence rules:

① **Idempodent laws:**

$P \land P \Leftrightarrow P$, $P \lor P \Leftrightarrow P$

② **Associative laws:**

$(P \land Q) \land R \Leftrightarrow P \land (Q \land R)$

$(P \lor Q) \lor R \Leftrightarrow P \lor (Q \lor R)$

③ **Commutative laws:**

$P \land Q \Leftrightarrow Q \land P$

$P \lor Q \Leftrightarrow Q \lor P$

④ **Demorgan's laws:**

$\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$

$\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$

⑤ **Distributive laws:** $(P \land Q) \lor R \Leftrightarrow (P \lor R) \land (Q \lor R)$

$(P \lor Q) \land R \Leftrightarrow (P \land R) \lor (Q \land R)$

$P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R)$

$P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)$

⑥ **Complement laws:**

$P \land \neg P \Leftrightarrow F$

$P \lor \neg P \Leftrightarrow T$

⑦ **Dominance laws:**

$P \lor T \Leftrightarrow T$

$P \land F \Leftrightarrow F$

⑧ **Identity laws:**

$P \land T \Leftrightarrow P$

$P \lor F \Leftrightarrow P$

⑨ **Absorption laws:**

$P \lor (P \land Q) \Leftrightarrow P$

$P \land (P \lor Q) \Leftrightarrow P$

(10) **Double negation law:**

$$\neg(\neg P) \Leftrightarrow P$$

(11) **Contrapositive law:**

$$P \to Q \Leftrightarrow \neg Q \to \neg P$$

(12) **Conditional as disjunction:**

$$P \to Q \Leftrightarrow \neg P \vee Q$$

(13) **Biconditional as conditional:**

$$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$$

(14) **Exportations laws:**

$$P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$$

i) show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivale

(1) p	(2) q	(3) r	(4) $q \wedge r$	(5) $p \vee (q \wedge r)$	(6) $p \vee q$	(7) $p \vee r$	(8) $(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	F	T	F	F	F	F	F
F	T	F	F	F	T	F	F
F	F	F	F	F	F	F	F

Here columns (5) and (8) are equivalent.

$\therefore$ $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Tautology:

A statement formula which is always true is called tautology.

Ex: $P \vee \neg P$

Contradiction:

A statement formula which is always false is called contradiction.

Ex: $P \wedge \neg P$

Contigency:

A statement formula which is neither tautology nor contradiction is called contigency.

Ex: $P \leftrightarrow Q$

1) Determine whether $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is a tautology.

$(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$	Reason
$\Leftrightarrow (\neg Q \wedge (\neg P \vee Q)) \rightarrow \neg P$	Conditional as disjunction
$\Leftrightarrow \neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	Conditional as disjunction
$\Leftrightarrow (Q \vee \neg(\neg P \vee Q)) \vee \neg P$	Demorgan's law, Double negation law
$\Leftrightarrow (Q \vee (P \wedge \neg Q)) \vee \neg P$	Demorgan's law, Double negation law
$\Leftrightarrow ((Q \vee P) \wedge (Q \vee \neg Q)) \vee \neg P$	Distributive law
$\Leftrightarrow ((Q \vee P) \wedge T) \vee \neg P$	Complement law
$\Leftrightarrow (Q \vee P) \vee \neg P$	Identity law
$\Leftrightarrow Q \vee (P \vee \neg P)$	Associative law
$\Leftrightarrow Q \vee T$	Complement law
$\Leftrightarrow T$	Dominance law

$\therefore (\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is a tautology.

2) Show that $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

Consider,

$(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))$	Reason
$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg(\neg Q \vee \neg R))$	Demorgans law, Double negation law
$\Leftrightarrow (P \vee Q) \wedge (P \vee (Q \wedge R))$	Demorgans law, Double negation law
$\Leftrightarrow (P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R))$	Distributive law
$\Leftrightarrow ((P \vee Q) \wedge (P \vee Q)) \wedge (P \vee R)$	Associative law
$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$ — (1)	Idempodent law

Consider,

$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$	Reason
$\neg(P \vee Q) \vee \neg(P \vee R)$	Demorgans law
$\neg((P \vee Q) \wedge (P \vee R))$ — (2)	Demorgans law

From (1) and (2)

$((P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$	Reasom
$\Leftrightarrow \quad T$	complement law

$\therefore ((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

3) Show that $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

$(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$	Reason
$\Leftrightarrow ((\sim p \wedge \sim q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$	Associative law
$\Leftrightarrow (\sim (p \vee q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$	Demorgans law
$\Leftrightarrow (\sim (p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$	Distributive law
$\Leftrightarrow (\sim (p \vee q) \wedge r) \vee ((p \vee q) \wedge r)$	Commutative law

$\Leftrightarrow$ [illegible] $\vee (p \vee q)) \wedge r$	distributive law
$\Leftrightarrow ((p \vee q) \vee \sim(p \vee q)) \wedge r$	Commutative law
$\Leftrightarrow T \wedge r$	complement law
$\Leftrightarrow r$	Identity law

$\therefore (\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$ is showed.

4) Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent without using truth table.

$\neg(p \vee (\neg p \wedge q))$	Reason
$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	Demorgans law
$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	Demorgans law, Double negation law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law
$\Leftrightarrow (p \wedge \neg p) \vee (\neg p \wedge \neg q)$	Commutative law
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Complement law
$\Leftrightarrow (\neg p \wedge \neg q) \vee F$	Commutative law
$\Leftrightarrow \neg p \wedge \neg q$	Identity law

$\therefore \neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent.

5) Show that the following implications without constructing the truth table.

$[(p \to q) \to q] \Rightarrow (p \vee q)$

$(p \to q) \to q$	Reason
$\Leftrightarrow \neg q \to (\neg(p \to q))$	Contrapositive law
$\Leftrightarrow \neg q \to (\neg(\neg q \to \neg p))$	Contrapositive law
$\Leftrightarrow q \vee (\neg(\neg q \to \neg p))$	Conditional as disjunction, Double negation law
$\Leftrightarrow q \vee (\neg(q \vee \neg p))$	Conditional as disjunction, Double negation law
$\Leftrightarrow q \vee (\neg q \wedge p)$	Demorgans law, Double negation law
$\Leftrightarrow (q \vee \neg q) \wedge (q \vee p)$	Distributive law
$\Leftrightarrow T \wedge (q \vee p)$	Complement law

$\Leftrightarrow q \vee p$	Identity law, commutative law
$\Leftrightarrow p \vee q$	Commutative law

$\therefore [(p \to q) \to q] \Rightarrow p \vee q$ ~~are~~ implication is True.

6) Prove that $(p \to q) \Rightarrow p \to (p \wedge q)$

$(p \to q) \to (p \to (p \wedge q))$

$(p \to q) \to (p \to (p \wedge q))$	Reason
$\Leftrightarrow (\neg p \vee q) \to (\neg p \vee (p \wedge q))$	Conditional as disjunction
$\Leftrightarrow \neg(\neg p \vee q) \vee (\neg p \vee (p \wedge q))$	condition as disjunction
$\Leftrightarrow (p \wedge \neg q) \vee ((\neg p \vee p) \wedge (\neg p \vee q))$	Demorgans law, distributive law, Double negation law.
$\Leftrightarrow (p \wedge \neg q) \vee (T \wedge (\neg p \vee q))$	Commutative law, Complement
$\Leftrightarrow (p \wedge \neg q) \vee ((T \wedge \neg p) \vee (T \wedge q))$	~~Identi~~ Distributive law
$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \vee (T \wedge q))$	Identity law
$\Leftrightarrow (p \wedge \neg q) \vee ((\neg p \vee T) \wedge (\neg p \vee q))$	Distributive law
$\Leftrightarrow (p \wedge \neg q) \vee (T \wedge (\neg p \vee q))$	Dominance law
$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \vee q)$	Identity law
$\Leftrightarrow (p \vee (\neg p \vee q)) \wedge (\neg q \vee (\neg p \vee q))$	Distributive law
$\Leftrightarrow ((p \vee \neg p) \vee q) \wedge ((\neg q \vee q) \vee \neg p)$	Commutative law, Associative
$\Leftrightarrow (T \vee q) \wedge (T \vee \neg p)$	Commutative law, Complement
$\Leftrightarrow T \wedge T$	Commutative law, Dominance law
$\Leftrightarrow T$	Idempodent law

$\therefore (p \to q) \to (p \to (p \wedge q))$ is a tautology

$\therefore (p \to q) \Rightarrow p \to (p \wedge q)$ is proved.

Converse:

Let $p \to q$ be the conditional proposition.

then $q \to p$ is called its converse.

Contrapositive

$\neg q \to \neg p$ is called its contrapositive

Inverse:

$\neg p \to \neg q$ is called inverse.

Remark:

① The conditional proposition and its contrapositive are logically equivalent.

That is. $p \to q \Leftrightarrow \neg q \to \neg p$

② The conditional proposition and its converse are not logically equivalent.

That is,

$p \to q$ and $q \to p$ are not logically equivalent.

③

1) Obtain converse, contrapositive and Inverse of "Team India wins whenever Dhoni is a Captain".

The given statement can be rewritten as, "If Dhoni is a captain then team India wins".

P: Dhoni is a Captain.

Q: Team India wins.

Converse: $Q \to P$

If Team India wins then Dhoni is a Captain.

Contrapositive: $\neg Q \to \neg P$

If Team India doesn't win then Dhoni is not a Captain.

Inverse: $\neg P \to \neg Q$

If Dhoni is not a captain then Team India doesn't win.

2) Determine the contrapositive of each statement:

$\neg Q \to \neg P$

(i) If x is less than 0, then x is not positive

(ii) If he has courage, he will win.

(iii) If John is a poet, then he is poor.

(iv) It is necessary to be Strong in order to be a sailor

(i) If x is positive, then x is not less than 0.

(ii) If he will not win, then he hasn't courage.

(iii) If John is not poor, then he is not a poet.

(iv) If he is not strong then he is not a sailor.

3) Find the contrapositive of the inverse of $P \to Q$

Sol:

* Inverse of $P \to Q$ is $\neg P \to \neg Q$

* contrapositive of $\neg P \to \neg Q$ is

$$\neg(\neg Q) \to \neg(\neg P)$$

$$Q \to P$$

4) Given the converse and the contrapositive of the implication "If it is raining, then I get wet.

P: It is raining

Q: I get wet

Coverse:

$Q \to P$

If I get wet then it is raining.

Contrapositive: $\neg Q \to \neg P$

If did not got wet then it is not raining.

Note: $P \to Q$

i) P implies Q

ii) Q if P

iii) P only if Q

iv) Q is necessary for P

v) P is sufficient for Q.

Formula:

$$P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Normal forms:

If we write the given statement formula in a particular form (in terms of $\wedge$, $\vee$ and $\neg$) then it is called Normal form. Here given statement formula and its normal form are equivalent.

Elementry Product:

* A product of the statement variables and their negation in a formula is called Elementry product.
* For an example, Let P and Q be any two atomic variables.
* Then possible elementry products are $P, Q, \neg P \wedge Q, Q \wedge \neg P, \neg Q \wedge P, Q \wedge \neg Q, P \wedge \neg P \wedge Q$.

Remarks:

⇒ A statement variable alone is and elementry product. Because 'P' can be written as, $P \wedge P$

Elementry Sum:

* A sum of the two statement variable and their negation is called Elementry Sum.
* For an example, Let P and Q be any two atomic variable.
* Then possible elementry sum are $P, Q, \neg P \vee Q, P \vee \neg Q, P \vee \neg P \vee Q$.

Disjunctive Normal forms (DNF)

A statement formula which is equivalent to a given formula and which consist of a sum of elementry products is called a DNF. of the given formula.

That is DNF = (Elementry product) ∨ (Elementry product) ∨ (") .... ∨ (Elementry product).

Conjuctive Normal forms (CNF)

A statement formula which is equivalent to a given formula and which consist of a product of elementry sums is called a CNF of the given formula.

That is CNF = (Elementry sum) ∧ (Elementry sum) ∧ ..... ∧ (Elementry sum).

1) Obtain DNF of $P \to [(P \to Q) \land \neg(\neg Q \lor \neg P)]$

$P \to [(P \to Q) \land \neg(\neg Q \lor \neg P)]$	Reason
$\Leftrightarrow \neg P \lor [(P \to Q) \land \neg(\neg Q \lor \neg P)]$	Conditional as disjunction
$\Leftrightarrow \neg P \lor [(\neg P \lor Q) \land (Q \land P)]$	Conditional as disjunction, Demorgan's law, double negation law
$\Leftrightarrow \neg P \lor [(\neg P \land (Q \land P)) \lor (Q \land (Q \land P))]$	Distributive law
$\Leftrightarrow \neg P \lor [((\neg P \land P) \land Q) \lor ((Q \land Q) \land P)]$	Commutative law, assodative law
$\Leftrightarrow \neg P \lor (F \land Q) \lor (P \land Q)$	Commutative law, complement law, Idempodent law
$\Leftrightarrow \neg P \lor F \lor (P \land Q)$	Dominance law
$\Leftrightarrow \neg P \lor (P \land Q)$	Identity law

$\Rightarrow \neg P \lor (P \land Q)$ which is required form in DNF

2) Obtain CNF of $P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

$P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$	Reason
$\Leftrightarrow \neg P \vee [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$	Conditional as disjunction
$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$	conditional as disjunction, Demorgans law, Double negation law
$\Leftrightarrow (\neg P \vee (\neg P \vee Q)) \wedge (\neg P \vee (Q \wedge P))$	Distributive law
$\Leftrightarrow ((\neg P \vee \neg P) \vee Q) \wedge [(\neg P \vee Q) \wedge (\neg P \vee P)]$	Associative law, Distributive law
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q) \wedge T$	commutative law, complement law
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q)$	Identity law

$\Rightarrow (\neg P \vee Q) \wedge (\neg P \vee Q)$ which is required form in CNF.

Minterms:

Let P and Q be 2 variables.

Then the minterms are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$

Maxterms:

Let P and Q be 2 variables.

Then the maxterms are $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, and $\neg P \vee \neg Q$.

Principle disjunctive Normal forms (PDNF) (Sum of products canonical form)

- For a given statement formula an equivalent formula consisting of disjunction of minterms only. is known as PDNF.

That is,
PDNF = (Minterms) v (Minterms) v . . . . . v (Minterms).

Principle conjuctive Normal forms (PCNF) (Product of sums canonical forms)

- For a given statement formula an equivalent formula [illegible] maxterms only is known as

That is,

PCNF = (Maxterms) $\wedge$ (Maxterms) $\wedge$ .... $\wedge$ (Maxterms).

Note:

PDNF	PCNF
$P \wedge T \Leftrightarrow P$	$P \vee F \Leftrightarrow P$
$P \vee \neg P \Leftrightarrow T$	$P \wedge \neg P \Leftrightarrow F$

1) Find the PDNF and PCNF of the statement $(\sim P \rightarrow R) \wedge (Q \leftrightarrow P)$

$(\sim P \rightarrow R) \wedge (Q \leftrightarrow P)$	Reason
$\Leftrightarrow (P \vee R) \wedge ((Q \wedge P) \vee (\neg Q \wedge \neg P))$	Conditional as disjunction. Biconditional as disjunction
$\Leftrightarrow (P \vee R) \wedge ((Q \wedge P) \vee \neg Q) \wedge ((Q \wedge P) \vee \neg P)$	Distributive law
$\Leftrightarrow (P \vee R) \wedge (Q \vee \neg Q) \wedge (P \vee \neg Q) \wedge (Q \vee \neg P) \wedge (P \vee \neg P)$	"
$\Leftrightarrow (P \vee R) \wedge T \wedge (P \vee \neg Q) \wedge (Q \vee \neg P) \wedge T$	Complementry law
$\Leftrightarrow (P \vee R) \wedge (P \vee \neg Q) \wedge (Q \vee \neg P)$	Identity law
$\Leftrightarrow (P \vee R \vee F) \wedge (P \vee \neg Q \vee F) \wedge (Q \vee \neg P \vee F)$	Identity law
$\Leftrightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge ((P \vee \neg Q) \vee (R \wedge \neg R)) \wedge ((Q \vee \neg P) \vee (R \wedge \neg R))$	Dominance law
$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (Q \vee \neg P \vee R) \wedge (Q \vee \neg P \vee \neg R)$	Distributive law
$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge \cancel{(P \vee \neg Q \vee \neg R)} \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$	Idempodent law, Commutative law

... required PCNF

PDNF:

$A : (P\vee Q\vee \neg R)\wedge(\neg P\vee \neg Q\vee R)\wedge(\neg P\vee \neg Q\vee \neg R)$

$\neg A : \neg(P\vee Q\vee \neg R)\vee\neg(\neg P\vee \neg Q\vee R)\vee\neg(\neg P\vee \neg Q\vee \neg R)$

$: (\neg P\wedge \neg Q\wedge R)\vee(P\wedge Q\wedge \neg R)\vee(P\wedge Q\wedge R)$

$: (\neg P\wedge \neg Q\wedge R)\vee(P\wedge Q\wedge \neg R)\vee(P\wedge Q\wedge R)$

which is required PDNF

2) Find the PDNF of the statement $(q\vee(p\wedge r))\wedge\sim((p\vee r)\wedge q)$

$=(p\wedge r)\vee(q\wedge r)$
$(p\vee q)\wedge$

$(q\vee(p\wedge r))\wedge\sim((p\vee r)\wedge q)$	Reason
$\Leftrightarrow (q\vee(p\wedge r))\wedge(\sim(p\vee r)\vee\sim q)$	Demorgan's law
$\Leftrightarrow (q\vee(p\wedge r))\wedge((\sim p\wedge\sim r)\vee\sim q)$	Demorgans law
$\Leftrightarrow (q\wedge((\sim p\wedge\sim r)\vee\sim q))\vee$ $((p\wedge r)\wedge((\sim p\wedge\sim r)\vee\sim q))$	Distributive law $p\wedge(q\vee r)=(p\wedge q)\vee(p\wedge r)$
$\Leftrightarrow (q\wedge(\sim p\wedge\sim r))\vee(q\wedge\sim q)\vee$ $((p\wedge r)\wedge(\sim p\wedge\sim r))\vee((p\wedge r)\wedge\sim q)$	Distributive law
$\Leftrightarrow (\sim p\wedge q\wedge\sim r)\vee F\vee F\vee(p\wedge\sim q\wedge r)$	Complementry law
$\Leftrightarrow (\sim p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge r)$	Identity law
which is required PDNF form.	

3) Find the PDNF and PCNF of the statement

$p\vee(\neg p\to(q\vee(\neg q\to r)))$

$p\vee(\neg p\to(q\vee(\neg q\to r)))$	Reason
$\Leftrightarrow p\vee(p\vee(q\vee(\neg q\to r)))$	Conditional as disjunction
$\Leftrightarrow p\vee(p\vee(q\vee(q\vee r)))$	Conditional as disjunction
$\Leftrightarrow p\vee(p\vee(q\vee r))$	Associative law, Idempodent law
$\Leftrightarrow p\vee q\vee r$	Associative law, Idempodent law
which is required PCNF form.	

PDNF:

$A: (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge$
$(p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg r)$

$\neg A: (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \wedge (p \wedge q \wedge \neg r)$
$(\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$

which is required PDNF.

4) Find the PDNF and PCNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$	Reason
$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$	Identity law
$\Leftrightarrow ((P \wedge Q) \wedge (R \vee \neg R)) \vee ((\neg P \wedge R) \wedge (Q \vee \neg Q)) \vee ((Q \wedge R) \wedge (P \vee \neg P))$	Complementry law
$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$	Distributive law
$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$ ~~v (P∧Q∧~~ Which is required PDNF form	commutative law

PCNF:

~~A~~ $(P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$\neg A: (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee R)$

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Theory of inference:

Rules:

$P, P \to Q \Rightarrow Q$	Modus Phonens
$\neg Q, P \to Q \Rightarrow \neg P$	Modus tollens
$\neg P, P \vee Q \Rightarrow Q$	Disjunctive syllogism
$P \to Q, Q \to R \Rightarrow P \to R$	Hypothetical syllogism (or) chain rule
$P, Q \Rightarrow P \wedge Q$	Simplification rule
$P \wedge Q \Rightarrow P, Q$	Simplification rule
$P, Q \Rightarrow P \vee Q$	Addition rule
$P \wedge \neg Q \Leftrightarrow \neg(P \to Q)$	Equivalence rule
$P \to Q \Leftrightarrow \neg Q \to \neg P$	Contrapositive rule

$\neg P \vee Q \Rightarrow P \to Q$

$P \wedge \neg Q \to \neg(P \to Q)$

1) Show that $J \wedge S$ logically follows from the premises $P \to Q, Q \to \neg R, R, P \vee (J \wedge S)$.

{1}	1) $P \to Q$	Rule P
{2}	2) $Q \to \neg R$	Rule P
{1,2}	3) $P \to \neg R$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{1,2}	4) $R \to \neg P$	Rule T, $P \to Q \Leftrightarrow \neg Q \to \neg P$
{5}	5) R	Rule P
{1,2,5}	6) $\neg P$	Rule T, $P, P \to Q \Rightarrow Q$
{7}	7) $P \vee (J \wedge S)$	Rule P
{1,2,5,7}	8) $J \wedge S$	Rule T, $\neg P, P \vee Q \Rightarrow Q$

2) Show that $R \to S$ can be derived from the premises $P \to (Q \to S), \neg R \vee P$ and Q.

{1}	1) R	Assumed Premises
{2}	2) $\neg R \vee P$	Rule P
{2}	3) $R \to P$	Rule T, $P \to Q \Leftrightarrow \neg P \vee Q$
{1,2}	4) P	Rule T, $P, P \to Q \Rightarrow Q$
{5}	5) $P \to (Q \to S)$	Rule P
{1,2,5}	6) $Q \to S$	Rule T, $P, P \to Q \Rightarrow Q$
{7}	7) Q	Rule P

{1,2,5,7}	8) S	Rule CP. $P, P \to Q \Rightarrow Q$

3) ~~Show~~ Prove that the premises $P \to Q$, $Q \to R$, $S \to \neg R$ and $P \wedge S$ are inconsistent. F

{1}	1) $P \to Q$	Rule P
{2}	2) $Q \to R$	Rule P
{1,2}	3) $P \to R$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{4}	4) $S \to \neg R$	Rule P
{4}	5) $R \to \neg S$	Rule T, $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1,2,4}	6) $P \to \neg S$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{1,2,4}	7) $\neg P \vee \neg S$	Rule P, $P \to Q \Leftrightarrow \neg P \vee Q$
{1,2,4}	8) $\neg(P \wedge S)$	Rule T, Demorgan's rule
{9}	9) $P \wedge S$	Rule P
{1,2,4,9}	10) $P \wedge S \wedge \neg(P \wedge S)$	Rule T, $P, Q \Rightarrow P \wedge Q$

We get false statement.

So $P \to Q$, $Q \to R$, $S \to \neg R$ and $P \wedge S$ are inconsisten

1) Find PDNF and PCNF of the Statement $(P \to (Q \wedge P)) \wedge (\neg P \to (\neg Q \wedge \neg R))$.

$(P \to (Q \wedge P)) \wedge (\neg P \to (\neg Q \wedge \neg R))$	Reason
$\Leftrightarrow (\neg P \vee (Q \wedge P)) \wedge (P \vee (\neg Q \wedge \neg R))$	Conditional as disjunction
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee P) \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$	Distributive law
$\Leftrightarrow (\neg P \vee Q) \wedge T \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$	Commutative law, Compleme
$\Leftrightarrow (\neg P \vee Q \vee F) \wedge (P \vee \neg Q \vee F) \wedge (P \vee F \vee \neg R)$	Identity law
$\Leftrightarrow ((\neg P \vee Q) \vee (R \wedge \neg R)) \wedge ((P \vee \neg Q) \vee (R \wedge \neg R)) \wedge ((P \vee \neg R) \vee (Q \wedge \neg Q))$	Complement law
$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$	Distributive law

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$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$ $\wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge \cancel{(P \vee \neg Q \vee \neg R)}$	Commutative law, Idempodent law
Which is required PCNF form	

<u>PDNF</u> :

$A : (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\neg A : (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

2) Find PDNF and PCNF of the statement $(Q \to P) \wedge (\neg P \wedge Q)$

$(Q \to P) \wedge (\neg P \wedge Q)$	Reason
$\Leftrightarrow (\neg Q \vee P) \wedge (\neg P \wedge Q)$	Conditional as disjunction
$\Leftrightarrow (\neg Q \wedge (\neg P \wedge Q)) \vee (P \wedge (\neg P \wedge Q))$	Distributive law
$\Leftrightarrow ((Q \wedge \neg Q) \wedge \neg P) \vee ((P \wedge \neg P) \wedge Q)$	Commutative law, Associative law
$\Leftrightarrow (F \wedge \neg P) \vee (F \wedge Q)$	
$\Leftrightarrow ((\neg Q \vee P) \wedge \neg P) \wedge Q$	Associative law
$\Leftrightarrow ((\neg Q \wedge \neg P) \vee (P \wedge \neg P)) \wedge Q$	Distributive law
$\Leftrightarrow ((\neg Q \wedge \neg P) \vee F) \wedge Q$	complementry law
$\Leftrightarrow (\neg Q \wedge \neg P \wedge Q)$	Identity law
$\neg P \wedge$	Commutative law, Complement law

4) Show that 'r' is the conclusion from the premises $p \to q$, $q \to r$, $\neg(p \wedge r)$, $p \vee r$, by indirect method.

{1}	1) $\neg r$	Additional premises
{2}	2) $p \vee r$	Rule P
{2}	3) $\neg r \to p$	Rule T, $P \to q \Leftrightarrow \neg P \vee q$
{1, 2}	4) p	Rule T, $P, P \to q \Rightarrow q$
{5}	5) $p \to q$	Rule P
{1, 2, 5}	6) q	Rule T, $P, P \to q \Rightarrow q$
{7}	7) $q \to r$	Rule P
{1, 2, 5, 7}	8) r	Rule T, $P, P \to q \Rightarrow q$
{1, 2, 5, 7}	9) $\neg r \wedge r$	Rule T, $P, q \Rightarrow P \wedge q$
{1, 2, 5, 7}	10) F	Rule T, $P \wedge \neg P \Leftrightarrow F$

5) Using indirect method, show that $R \to \neg Q$, $R \vee S$, $S \to \neg Q$, $P \to Q \Rightarrow \neg P$

{1}	1) $\neg(\neg P)$	Additional premises
{1}	2) P	Rule T, Double negation law
{3}	3) $P \to Q$	Rule P
{1, 3}	4) Q	Rule T, $P, P \to Q \Rightarrow Q$
{5}	5) $R \to \neg Q$	Rule P
{5}	6) $Q \to \neg R$	Rule T, Contrapositive law
{1, 3, 5}	7) $\neg R$	Rule T, $P, P \to Q \Rightarrow Q$
{8}	8) $R \vee S$	Rule P
{8}	9) $\neg R \to S$	Rule T, $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 3, 5, 8}	10) S	Rule T, $P, P \to Q \Rightarrow Q$
{11}	11) $S \to \neg Q$	Rule P
{1, 3, 5, 8, 11}	12) $\neg Q$	Rule T, $P, P \to Q \Rightarrow Q$

{1,3,5,8,11}	F	Rule T, $P \wedge \neg P \Leftrightarrow F$

6) Show that the following argument is valid

(If 7 is less than 4, then 7 is not a Prime number.) 7 is a prime number. Therefore 7 is not less than 4

P: 7 is less than 4

q: 7 is a prime number

Premises:

$P \to \neg q$

q

Conclusion:

$\neg P$

{1}	1) $P \to \neg q$	Rule P
{1}	2) $q \to \neg P$	Rule T, Contrapositive law
{3}	3) q	Rule P
{1,3}	4) $\neg P$	Rule T, $P, P \to Q \Rightarrow Q$

7) Show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday", "We will go swimming only if it is sunny", "If we do not swimming, then we will take canoe trip" & "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset".

P: It is sunny this afternoon.

Q: It is colder than yesterday

R: We will go swimming

S: We will take a conoe trip

T: We will be home by sunset.

Premises:

$\neg P \wedge Q$, $R \to P$, $\neg R \to S$, $S \to T$

Conclusion:

T

{1}	1) $R \to P$	Rule P
{2}	2) $\neg R \to S$	Rule P
{2}	3) $\neg S \to R$	Rule T, $P \to R \Leftrightarrow \neg R \to \neg P$
{1,2}	4) $\neg S \to P$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{1,2}	5) $\neg P \to S$	Rule T, $P \to Q \Leftrightarrow \neg Q \to \neg P$
{6}	6) $S \to T$	Rule P
{1,2,6}	7) $\neg P \to T$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{8}	8) $\neg P \wedge Q$	Rule P
{8}	9) $\neg P$	Rule T, $P \wedge Q \Rightarrow P$
{1,2,6,8}	10) T	Rule T, $P, P \to Q \Rightarrow Q$

8) Show that " It rained" is a conclusion Obtained from the statements. " If it does not rained or if there is no traffic dislocation, then the sports day will be held and the cultural program will go on." " If the sports day is held, ther trophy will be awarded" and " The trophy was not awarded".

P: It is rained

Q : There is traffic dislocation

R : The sports day will be held

S : The cultural program will go on

T : The trophy will be awarded

Premises:

$(\neg P \vee \neg Q) \to (R \wedge S)$, $R \to T$, $\neg T$.

Conclusion:

P

{1}	1) $(\neg P \vee \neg Q) \to (R \wedge S)$	Rule P
{1}	2) $(P \wedge Q) \vee (R \wedge S)$	Rule T, $P \to Q \Leftrightarrow \neg P \vee Q$
{1}	3) $P \vee R$	Rule T, $P \wedge Q \Rightarrow P, Q$
{1}	4) $\neg P \to R$	Rule T, $P \to Q \Leftrightarrow \neg P \vee R$
{5}	5) $R \to T$	Rule P

{1,5}	6) $\neg P \to T$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
{1,5}	7) $\neg T \to P$	Rule T, $P \to Q \Leftrightarrow \neg Q \to \neg P$
{8}	8) $\neg T$	Rule P
{1,5,8}	9) P	Rule P, $P, P \to Q \Rightarrow P$

Universal Quantifier $\forall x$ or (x)

(i) For all x

(ii) For every x

(iii) For each x

(iv) Everything x is such that

(v) Each thing x is such that

Existential Quantifier $\exists x$

(i) For some x

(ii) Some x such that

(iii) there exists on x such that

(iv) there is an x such that

(v) there is atleast one x such that

Universal Generalization (UG):

$A(y) \Rightarrow (x)\, A(x)$

Existential Generalization (EG):

$A(y) \Rightarrow (\exists x)\, A(x)$

Universal specification (US):

$(x)\, A(x) \Rightarrow A(y)$

Existential specification (ES):

$(\exists x)\, A(x) \Rightarrow A(y)$

1) Prove that ~~there is~~ $(\exists x)(P(x) \land Q(x)) \Rightarrow (\exists x)P(x) \land (\exists x)\, Q(x)$.

{1}	1) $(\exists x)(P(x) \land Q(x))$	Rule P
{1}	2) $P(y) \land Q(y)$	Rule ES
{2}	3) $P(y)$	Rule T, $P \land Q \Rightarrow P$
{3}	4) $(\exists x) P(x)$	Rule EG
{2}	5) $Q(y)$	Rule T, $P \land Q \Rightarrow Q$
{5}	6) $(\exists x) Q(x)$	Rule EG
{4,6}	7) $(\exists x) P(x) \land (\exists x) Q(x)$	Rule T, $P, Q \Rightarrow P \land Q$

2) Using the indirect method, show that

$(x)(P(x) \lor Q(x)) \Rightarrow$

3) Show that the premises "One Student in this class knows how to write programs in Java" and "Everyone who knows how to write programs in Java can get a high paying job" imply the conclusion "Someone in this class can get a high paying job".

$P(x)$: x is in this class

$Q(x)$: x knows how to write programs in Java

$R(x)$: x can get a high paying job.

<u>Premises:</u>

$(\exists x)(P(x) \land Q(x))$

$(x)(Q(x) \rightarrow R(x))$

<u>Conclusion:</u>

$(\exists x)(P(x) \land R(x))$

{1}	1) $(\exists x)(P(x) \land Q(x))$	Rule P
{1}	2) $P(y) \land Q(y)$	Rule ES
{3}	3) $(x)(Q(x) \rightarrow R(x))$	Rule P
{3}	4) $Q(y) \rightarrow R(y)$	Rule US
{2}	5) $P(y)$	Rule T, $P \land Q \Rightarrow P$
{2}	6) $Q(y)$	Rule T, $P \land Q \Rightarrow Q$
{4,6}	7) $R(y)$	Rule T, $P, P \rightarrow Q \Rightarrow Q$
{5,7}	8) $P(y) \land R(y)$	Rule T, $P, Q \Rightarrow P \land Q$
{8}	9) $(\exists x)(P(x) \land R(x))$	Rule EG

4) Establish the validity of the following argument: "All integers are rational numbers", "Some integers are powers of 2". Therefore "some rational numbers are powers of 2".

$P(x)$: x is an integer

$Q(x)$: x is rational

$R(x)$: x is power of 2.

Premises:

$(x)(P(x) \to Q(x))$

$(\exists x)(P(x) \land R(x))$

Conclusion

$(\exists x)(Q(x) \land R(x))$

{1}	1) $(x)(P(x) \to Q(x))$	Rule P
{1}	2) $P(y) \to Q(y)$	Rule US
{3}	3) $(\exists x)(P(x) \land R(x))$	Rule P
{3}	4) $P(y) \land R(y)$	Rule ES
{4}	5) $P(y)$	Rule T, $P \land Q \Rightarrow P$
{4}	6) $R(y)$	Rule T, $P \land Q \Rightarrow Q$
{2,5}	7) $Q(y)$	Rule T, $P, P \to Q \Rightarrow Q$
{6,7}	8) $Q(y) \land R(y)$	Rule T, $P, Q \Rightarrow P \land Q$
{8}	9) $(\exists x)(Q(x) \land R(x))$	Rule EG

5) Show that the following argument is valid, "Every micro computer has a serial interface port". "Some micro computer have a parallel port. Therefore some, micro computers have both serial interfaces port and parallel port".

$P(x)$: x is a micro computer

$Q(x)$: x has serial interface port

$R(x)$: x has parallel port.

Premises.

$(x)(P(x) \to Q(x))$

$(\exists x)(P(x) \land R(x))$

Conclusion :

$(\exists x)($[illegible]$\land Q(x) \land R(x))$

{1}	1) $(x)(P(x) \rightarrow Q(x))$	Rule P
{1}	2) $P(y) \rightarrow Q(y)$	Rule US
{3}	3) $(\exists x)(P(x) \wedge R(x))$	Rule P
{3}	4) $P(y) \wedge R(y)$	Rule ES
{4}	5) $P(y)$	Rule T, $P \wedge Q \Rightarrow P$
{4}	6) $R(y)$	Rule T, $P \wedge Q \Rightarrow Q$
{2,5}	7) $Q(y)$	Rule T, $P, P \rightarrow Q \Rightarrow Q$
{6,7}	8) $Q(y) \wedge R(y)$	Rule T, $P, Q \Rightarrow P \wedge Q$
{5,8}	9) $P(y) \wedge Q(y) \wedge R(y)$	Rule T $P, Q \Rightarrow P \wedge Q$
{9}	10) $(\exists x)(P(x) \wedge Q(x) \wedge R(x))$	Rule EG

6) Show that the premises "A student in this class has not read the book" and "Everyone in this class passed the exam" imply the conclusion "Someone who passed the exam has not read the book".

$P(x)$: x in this class

$Q(x)$: x has ~~not~~ not read the book.

$R(x)$: x passed the exam.

Premises:

$(\exists x)(P(x) \wedge Q(x))$

$(x)(P(x) \rightarrow R(x))$

Conclusion:

$(\exists x)(R(x) \wedge Q(x))$

{1}	1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
{1}	2) $P(y) \wedge Q(y)$	Rule ES
{2}	3) $P(y)$	Rule T, $P \wedge Q \Rightarrow P$
{2}	4) $Q(y)$	Rule T, $P \wedge Q \Rightarrow Q$
{5}	5) $(x)(P(x) \rightarrow R(x))$	Rule P
{5}	6) $P(y) \rightarrow R(y)$	Rule US
{3,6}	7) $R(y)$	Rule T, $P, P \rightarrow Q \Rightarrow Q$
{4,7}	8) $R(y) \wedge Q(y)$	Rule T, $P, Q \Rightarrow P \wedge Q$
{8}	9) $(\exists x)(R(x) \wedge Q(x))$	Rule EG

7) Use the indirect method to prove that the conclusion $\exists x\, Q(x)$ follows from the premises $\forall x\,(P(x) \to Q(x))$ and $\exists x\, P(x)$.

{1}	1) $\neg(\exists x)\, Q(x)$	Assumed premises
{1}	2) $(x)\, \neg Q(x)$	Rule T, Demorgans law
{2}	3) $\neg Q(y)$	Rule US
{4}	4) $(x)(P(x) \to Q(x))$	Rule P
{4}	5) $P(y) \to Q(y)$	Rule US
{6}	6) $(\exists x)\, P(x)$	Rule P
{6}	7) $P(y)$	Rule ES
{5,7}	8) $Q(y)$	Rule T, $P, P \to Q \Rightarrow Q$
{3,8}	9) $Q(y) \wedge \neg Q(y)$	Rule T, $P, Q \Rightarrow P \wedge Q$

Hence the final statement value is false

$\forall x\,(P(x) \to Q(x))$ and $\exists x\, P(x) \Rightarrow \exists x\, Q(x)$.

8) Show that the following this statement are logically equivalent "It is not true that all comedians are funny" and "There are some comedians who are not funny".

$P(x)$: x is a comedian

$Q(x)$: x is funny.

Premises:

$(x)\,(P(x) \to Q(x))$

$\neg (x)(P(x) \to Q(x))$

$(\exists x)\, \neg(P(x) \to Q(x))$

Conclusion:

$(\exists x)\ (P(x) \wedge \neg Q(x))$

{1}	(1) $(\exists x)\, \neg(P(x) \to Q(x))$	Rule P
{1}	2) $\neg(P(y) \to Q(y))$	Rule ES
{2}	3) $\neg(\neg P(y) \vee Q(y))$	Rule T, $P \to Q \Leftrightarrow \neg P \vee Q$
	4) $P(y) \wedge \neg Q(y)$	Rule T, Demorgan's law
{4}	5) $(\exists x)(P(x) \wedge \neg Q(x))$	Rule EG

9) Using proof by contradiction, prove that $\sqrt{2}$ is a irrational.

Assume $\sqrt{2}$ is rational.

that is $\sqrt{2} = \frac{p}{q}$ $p, q \in z$, $q \neq 0$, p & q have no common divisors.

$$2 = \frac{p^2}{q^2}$$

$$p^2 = 2q^2 \quad \text{--- (1)}$$

$\Rightarrow p^2$ is even $\Rightarrow p$ is even

Take $p = 2m$ where 'm' is any integer.

$\therefore (1) \Rightarrow (2m)^2 = 2q^2$

$$\frac{4m^2}{2} = q^2$$

$$2m^2 = q^2$$

$\Rightarrow q^2$ is even $\Rightarrow q$ is even

Take $q = 2n$ where 'n' is any integer.

$\therefore$ p and q are even numbers. p and q have a common divisor 2.

Which is contradiction to our assumption.

$\therefore \sqrt{2}$ is irrational.

---

10) Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x))$ $\Rightarrow (x)(P(x) \rightarrow R(x))$

Premises:

$(x)(P(x) \rightarrow Q(x))$

$\wedge$

$(x)(Q(x) \rightarrow R(x))$

Conclusion:

$(x)(P(x) \rightarrow R(x))$

{1}	1) $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x))$	Rule P,
{1}	2) $(x)(P(x) \rightarrow Q(x))$	Rule T, $P \wedge Q \Leftrightarrow P, Q$
{1}	3) $(x)(Q(x) \rightarrow R(x))$	Rule T, $P \wedge Q \Leftrightarrow P, Q$
{2}	4) $P(y) \rightarrow Q(y)$	Rule US
{3}	5) $Q(y) \rightarrow R(y)$	Rule US
{4,5}	6) $P(y) \rightarrow R(y)$	Rule T, $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow$
{6}	7) $(x)(P(x) \rightarrow R(x))$	Rule UG.

Unit-2

COMBINATORICS

Principle of mathematical induction

A statement $p(n)$ true for all natural numbers.

Step 1 : We must prove that $P(1)$ is true.

Step 2 : By assuming $P(k)$ is true, we must prove that $P(k+1)$ is also true.

Note:

* Step 1 is known as the basis step.
* Step 2 is known as the inductive step.

1) Using mathematical induction prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Let $P(n): 1^2+2^2+3^2+\dots\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: $P(1): 1^2 = \frac{1(2)(3)}{6} = 1$

$1 = 1$

$\therefore P(1)$ is true.

Step 2: Assume $P(k): 1^2+2^2+\dots\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

Claim: $P(k+1)$ is true.

$$1^2+2^2+\dots\dots+(k+1)^2 = 1^2+2^2+\dots\dots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$\therefore P(k+1)$ is true.

Hence by mathematical induction $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all n.

---

2) Using mathematical induction prove that

$$\sum_{m=0}^{n} 3^m = \frac{3^{n+1}-1}{2}$$

Let $p(n) : 3^0 + 3^1 + \dots + 3^n = \frac{3^{n+1}-1}{2}$

Step 1: $P(0) : 3^0 = \frac{3^1-1}{2} = \frac{2}{2} = 1$

$1 = 1$

$\therefore p(0)$ is true.

Step 2: Assume $P(k) : 3^0+3^1+\dots+3^k = \frac{3^{k+1}-1}{2}$ is true

claim: $P(k+1)$ is true.

$3^0 + \dots + 3^{k+1} = 3^0+3^1+\dots+3^k+3^{k+1}$

$= \frac{3^{k+1}-1}{2} + 3^{k+1}$

$$= \frac{1(3^{k+1}-1)+(2(3^{k+1}))}{2}$$

$$= \frac{3(3^{k+1})-1}{2}$$

$$= \frac{3^{k+2}-1}{2}$$

$$= \frac{3^{(k+1)+1}-1}{2}$$

$\therefore$ P(k+1) is true.

Hence by mathematical induction $\sum_{m=0}^{n} 3^m = \frac{3^{n+1}-1}{2}$ is true for all n.

---

3) Using mathematical induction prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Let P(n) : $\frac{1}{1(2)} + \frac{1}{2(3)} + \dots \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1: P(1) : $\frac{1}{1(2)} = \frac{1}{1+1} = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{2}$$

$\therefore$ P(1) is true.

Step 2: Assume P(k): $\frac{1}{1(2)} + \frac{1}{2(3)} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ is true.

Claim: P(k+1) is true.

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \dots + \frac{1}{(k+1)((k+1)+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)}$$

(2n+1)

is true.

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+1)(k+2) + (k+1)}{(k+1)(k+1)(k+2)}$$

$$= \frac{\cancel{(k+1)}\left[k(k+2)+1\right]}{\cancel{(k+1)}(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{(k+1)}{((k+1)+1)}$$

$\therefore$ $p(k+1)$ is true.

Hence by mathematical induction

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$ is true for all n

4) Using mathematical induction prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

 : 133 divides $11^{n+1} + 12^{2n-1}$ where n is a positive integer.

Step 1: $P(1): 11^2 + 12 = 121 + 12 = 133$

$\therefore$ 133 divides 133

$\therefore$ P(1) is true.

Step 2: Assume P(k): 133 divides $11^{k+1} + 12^{2k-1}$ is true.

claim: P(k+1) is true.

$11^{k+1+1} + 12^{2(k+1)-1} = 11^{k+2} + 12^{2k+2-1}$

$= 11^{k+1} \cdot 11 + 12^{2k-1} \cdot 12^2$

$= 11^{k+1} \cdot 11 + 11 \cdot 12^{2k-1} - 11 \cdot 12^{2k-1} + 12^{2k-1} \cdot 12^2$

$= 11(11^{k+1} + 12^{2k-1}) + 12^{2k-1}(12^2 - 11)$

$= 11(11^{k+1} + 12^{2k-1}) + 12^{2k-1}(144 - 11)$

$= 11(11^{k+1} + 12^{2k-1}) + 133(12^{2k-1})$

Here 133 divides $11^{k+1} + 12^{2k-1}$ ($\because$ our assumption)

and 133 divides $133(12^{2k-1})$ also.

$\therefore$ 133 divides $11^{k+2} + 12^{2(k+1)-1}$, Here P(k+1) is true

$\therefore$ By mathematical induction 133 divides $11^{n+1} + 12^{2n-1}$ where n is any positive integer.

5) Using mathematical induction to show that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Let P(n): $n^2 - 1$ is divisible by 8 where n is an odd positive integer.

Step 1: P(1): $1^2 - 1 = 1 - 1 = 0$ is divisible by 8.

$\therefore$ P(1) is true.

Step 2: Assume P(k): $k^2 - 1$ is divisible by 8 is true.

Claim: P(k+2) is true.

$(k+2)^2 - 1 = k^2 + 4k + 4 - 1$

$= k^2 - 1 + 4(k+1)$

Here k^2-1 is divisible by 8 ($\because$ our assumption)

and $k+1$ is an even number.

$\therefore 4(k+1)$ is divisible by 8.

Hence $(k+2)^2 - 1$ is divisible by 8.

$\therefore P(k+2)$ is true.

$\therefore$ By mathematical induction n^2-1 is divisible by 8 where n is an odd positive integer.

---

6) Prove that $8^n - 3^n$ is a multiple of 5.

Let $P(n): 8^n - 3^n$ is a multiple of 5.

Step 1: $P(1): 8^1 - 3^1 = 8-3 = 5$ is a multiple of 5.

$\therefore P(1)$ is true.

Step 2: Assume $P(k) = 8^k - 3^k$ is a multiple of 5 is true

claim: $P(k+1)$ is true.

$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$

$= 8^k \cdot 8 - 8 \cdot 3^k + 8 \cdot 3^k - 3^k \cdot 3$

$= 8(8^k - 3^k) + 3^k(8-3)$

$= 8(8^k - 3^k) + 5 \cdot 3^k$

Here $8^k - 3^k$ is a multiple of 5. ($\because$ by our assumption)

and $5 \cdot 3^k$ is also multiple of 5.

$\therefore 8^{k+1} - 3^{k+1}$ is a multiple of 5 $\therefore P(k+1)$ is true.

$\therefore$ By mathematical induction $8^n - 3^n$ is a multiple of 5 where n is any integer.

---

7) Show that $n < 2^n$.

Let $P(n): n < 2^n$

Step 1 : $P(1): 1 < 2^1$

$1 < 2$

$\therefore P(1)$ is true.

Step 2: Assume $P(k): k < 2^k$ is true.

Claim ~~Step 3~~ : $P(k+1)$ is true.

$$k < 2^k$$

$$k+1 < 2^k + 1$$

$$< 2^k + 2^k \quad (\because 1 < 2^k)$$

$$< 2 . 2^k$$

$$< 2^{k+1}$$

$\therefore k+1 < 2^{k+1}$

$\therefore P(k+1)$ is true.

$\therefore$ By mathematical induction $n < 2^n$ for all n.

8) Show that $2^n < n!$ for all $n \geq 4$.

Let $P(n): 2^n < n!$ for all $n \geq 4$.

Step 1: $P(4): 2^4 < 4!$

$16 < 24$

$\therefore P(4)$ is true.

Step 2: Assume $P(k): 2^k < k!$ is true.

Claim: $P(k+1)$ is true.

$$2^k < k!$$

$$2 . 2^k < 2k!$$

$$2^{k+1} < 2k!$$

$$2^{k+1} < (k+1)k! \quad (\because 2 < k+1)$$

$\therefore 2^{k+1} < (k+1)!$

$\therefore$ By mathematical Induction $2^n < n!$ for all $n \geq 4$.

9) Prove that $3^n + 7^n - 2$ is divisible by 8 for $n \geq 1$.

Let $P(n): 3^n + 7^n - 2$ is divisible by 8 for $n \geq 1$.

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Step 1 : $P(1) : 3^1 + 7^1 - 2 = 3 + 7 - 2 = 10 - 2 = 8$; $\frac{8}{8} = 1$ is divisible by 8.

$\therefore P(1)$ is true.

Step 2 : Assume $P(k) = 3^k + 7^k - 2$ is divisible by 8 for $n \geq 1$. is true.

$P(k+1)$ is true.

$$3^{k+1} + 7^{k+1} - 2 = 3^k \cdot 3 + 7^k \cdot 7 - 2$$

$$= 3^k \cdot 3 + 3 \cdot 7^k - 3 \cdot 2 - 3 \cdot 7^k + 3 \cdot 2 + 7^k \cdot 7$$

$$= 3(3^k + 7^k - 2) - 3 \cdot 7^k + 6 + 7^k \cdot 7 - 2$$

$$= 3(3^k + 7^k - 2) + 4 \cdot 7^k + 4$$

$$= 3(3^k + 7^k - 2) + 4(7^k + 1)$$

Here, $3^k + 7^k - 2$ is divisible by 8 and $4(7^k + 1)$ is also divisible by 8.

$\therefore 3^{k+1} + 7^{k+1} - 2$ is divisible by 8, so, $P(k+1)$ is true

$\therefore$ By mathematical induction, $3^{k+1} + 7^{k+1} - 2$ is divisible by 8 where $n \geq 1$.

Strong Induction

In this we use the basis step as before but we use a different inductive step.

We assume the $P(j)$ is true for $j = 1, 2, \ldots k$ and show that $P(k+1)$ must also be true, based on this assumption. This is called strong induction (2nd induction principle of mathematical induction)

1) Using mathematical induction prove that every integer $n \geq 2$ is either a prime number our product of prime number.

Proof: Every integer $n \geq 2$ is either a prime number or product of prime numbers.

Step 1 : $P(2)$: 2 is prime

$\therefore P(2)$ is true.

Step 2 : Assume $P(j)$ is true for all positive integer j with $j \le k$.

Claim : $P(k+1)$ is ~~prime~~ true.

Case (i) : $(k+1)$ is prime.

$\therefore P(k+1)$ is true.

Case (ii) : $k+1$ is composite.

2×2, 2×3, ab, ab, $k+1 = ab$, $2 \le a < b < k+1$

$\therefore k+1$ can be written as product of two positive integers 'a' and 'b' such that $2 \le a < b < k+1$.

$\Rightarrow$ 'a' and 'b' can be written as product of prime numbers. ($\because$ by our assumption)

$\therefore k+1$ can be written as product of primes.

$\therefore P(k+1)$ is true.

$\therefore$ By mathematical induction every integer $n \ge 2$ is either a prime number or product of prime numbers.

Pigeon hole Principle

If '(n+1)' pigeon occupies 'n' holes then atleast one hole has more than one pigeon.

Generalized pigeon hole principle

If 'm' pigeon occupies 'n' holes $(m > n)$ then at least one hole has more than $\left[\frac{m-1}{n}\right] + 1$ pigeon.

Here $[x]$ denotes the greatest integer $\le x$, which is a real number.

1) Show that among 100 people atleast 9 of them were born in the same month.

Number of people = 100 = No. of pigeon = m

Number of month = 12 = No. of holes = n

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{100-1}{12}\right]+1$$

$$= \left[\frac{99}{12}\right]+1$$

$$= 8+1$$

$$= 9$$

∴ Atleast 9 of them were born in the same month.

2) Show that if 7 colours are used to paint 50 bicycles, atleast 8 bicycles will be the same colour.

Number of bicycles = 50 = No. of pigeon = m

Number of colours = 7 = No. of holes = n

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{50-1}{7}\right]+1$$

$$= \left[\frac{49}{7}\right]+1$$

$$= 7+1$$

$$= 8$$

∴ Atleast 8 bicycles will be the same colour.

Permutation:

* Each of them different arrangements which can be made by taking some (or) all of a number of things at a time. This called a Permutation.

⇒ The number of permutations of 'n' things taken 'r' at a time is denoted by 'nP_r'

⇒ That is $nP_r = \frac{n!}{(n-r)!}$

Combination

* Each of the different groups (or) selections which can be made by taking some (or) all of a number of thing's at a time. This called a combination

⇒ It is denoted by 'nC_r'

⇒ That is, $nC_r = \frac{n!}{r!(n-r)!}$

1) In how many ways can letters of the word "INDIA" be arranged?

No.of ways $= \frac{5!}{2!}$

$= \frac{1\times2\times3\times4\times5}{1\times2}$

$= 60$

2) In how many ways can six persons occupy 8 vacant seats?

No.of. ways $= 6! \times 5! \times 4!$

$= 2,073,600.$

3) In how many ways can 5 person be selected from among 10 persons?

$\boxed{nC_r = \frac{n!}{r!(n-r)!}}$

No.of ways $= \frac{10!}{5!(10-5)!}$

$= \frac{10!}{5!\times5!}$

$= 252$

4) In how many ways can all the letters in "MATHEMATICAL" be arranged?

$$\text{No. of ways} = \frac{12!}{2! \times 3! \times 2!}$$

$$= 19,958,400.$$

5) A box contains '6' white balls and '5' red balls. Find the number of ways four balls can be drawn from the box.

i) They can be of any colour
ii) Two must be white and two red.
iii) They must all be of the same colour

Given:

6 - white balls

5 - red balls

(i) The no. of ways they can be of any colour

$$11C_4 = 330.$$

(ii) The no. of ways two must be white and two red

This and that

$$6C_2 \underset{\text{and}}{\times} 5C_2 = 150$$

(iii) The no. of ways they must all be of the same colour

This (or) that

$$6C_4 \underset{\text{or}}{+} 5C_4 = 20.$$

6) How many permutations can be made out of the letters of the word "BASIC". ~~How many~~ How many of these i) ~~How~~ Begin with B

ii) End with C

(iii) 'B' ~~and~~ or 'C' occupy the End position

No. of ways the word "Basic" can be arranged

$$= 5! = 120.$$

i) No. of ways the word begin with B.

$$= \boxed{1!} \times 4! \times 3! \times 2! \times 1!$$ rem 4, rem 3, rem 2, rem 1 | competition.

$$= 288$$

(ii) No. of ways the word end with 'c'.

$$= 4! \times 3! \times 2! \times 1! \times \boxed{1!}\, c$$

$$= 288.$$

(iii)
Case (i): End with B.

No. of ways the word end with B $= 4! \times 3! \times 2! \times 1! \times 1!$
$= 288$

Case (ii): End with c.

No. of ways the word end with c $= 4! \times 3! \times 2! \times 1! \times 1!$
$= 288.$

$\therefore$ No. of ways the word end with 'B' or 'C' $= 288 + 288$
$= 576.$

Recurrence Relation

- A linear homogeneous recurrence relation of degree 'k' with constants coefficient is of the form $a_n = c_1 a_{n+1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ where, $c_1, c_2, \dots\dots c_k$ are real numbers and $c_k \neq 0$.

- A linear non-homogeneous recurrence relation is of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$ where, $c_1, c_2, \dots, c_k$ are real number and $F(n)$ is the function of n.

Results

Case (i):

Roots are real and distinct say $r_1, r_2, \dots r_n$.

Then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_n r_n^n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are constants

Case (ii):

Roots are real and equal say $r_1 = r_2 = \dots = r_n$

Then,

$$a_n = \alpha_1 r_1^n + n\alpha_2 r_2^n + n^2\alpha_3 r_3^n + \dots + n^n \alpha_n r_n^n.$$

Case (iii):

Roots are complex and conjugates ~~say~~

$$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$$

---

1) Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ & $a_1 = 7$.

Given $a_n = a_{n-1} + 2a_{n-2}$ and $a_0 = 2, a_1 = 7$.

$$a_n - a_{n-1} - 2a_{n-2} = 0$$

Characteristic equation:

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0$$

$r+1=0$	$r-2=0$
$r=-1$	$r=2$

-2 -1 $\frac{-2}{r}$ $\frac{1}{r}$

$\therefore r = -1, 2$

$\therefore$ The solution is,

$$a_n = \alpha_1(-1)^n + \alpha_2(2)^n$$

$a_0 = 2$

$$a_0 = \alpha_1(-1)^0 + \alpha_2(2)^0 = 2$$

$$\Rightarrow \alpha_1 + \alpha_2 = 2 \quad \text{--- (1)}$$

$a_1 = 7$

$$a_1 = \alpha_1(-1)^1 + \alpha_2(2)^1 = 7$$

$$\Rightarrow -\alpha_1 + 2\alpha_2 = 7 \quad \text{--- (2)}$$

$(1)+(2) \Rightarrow 3\alpha_2 = 9$

$$\boxed{\alpha_2 = 3}$$

substitude α_2 value in (1),

$\alpha_1 + 3 = 2$

$\boxed{\alpha_1 = -1}$

$$\boxed{\therefore a_n = (-1)(-1)^n + 3(2)^n}$$

2) Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

Given : $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 6$.

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Characteristic equation:

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$$\therefore r = 3, 3$$

$\therefore$ The solution is,

$$a_n = \alpha_1 (3)^n + n\alpha_2 (3)^n$$

$a_0 = 1$

$$a_0 = \alpha_1 (3)^0 + 0 = 1$$

$$\boxed{\alpha_1 = 1}$$

$a_1 = 6$

$$a_1 = \alpha_1 (3)^1 + 1(\alpha_2)(3)^1 = 6$$

$$\Rightarrow 3\alpha_1 + 3\alpha_2 = 6$$

$$\Rightarrow 3(1) + 3\alpha_2 = 6 \qquad [\because \alpha_1 = 1]$$

$$3\alpha_2 = 6 - 3$$

$$3\alpha_2 = 3$$

$$\boxed{\alpha_2 = 1}$$

$$\boxed{\therefore a_n = (3)^n + n(3)^n}$$

---

3) Solve $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2, a_1 = 5$ & $a_2 =$

Given: $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2, a_1 = 5, a_2 = 15$

Characteristic equation:

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0.$$

$$r^3 - 6r^2 + 11r - 6 = 0.$$

	1	-6	11	-6
$r = 1$	0	1	-5	6
	1	-5	6	0

$$(r-3)(r-2)=0$$

$$r=3,2$$

$$r=1,2,3$$

The solution is

$$a_n = \alpha_1(1)^n + \alpha_2(2)^n + \alpha_3(3)^n$$

$\underline{a_0=2}$

$$a_0 = \alpha_1(1)^0 + \alpha_2(2)^0 + \alpha_3(3)^0 = 2$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2 \quad -(1)$$

$\underline{a_1=5}$

$$a_1 = \alpha_1(1)^1 + \alpha_2(2)^1 + \alpha_3(3)^1 = 5$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \quad -(2)$$

$\underline{a_2=15}$

$$a_2 = \alpha_1(1)^2 + \alpha_2(2)^2 + \alpha_3(3)^2 = 15$$

$$\Rightarrow \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \quad -(3)$$

(2) − (1)

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

$$(-)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$\alpha_2 + 2\alpha_3 = 3 \quad -(4)$$

(3) − (2)

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \quad (-)$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$2\alpha_2 + 6\alpha_3 = 10 \quad -(5)$$

(5) − 2(4)

$$2\alpha_2 + 6\alpha_3 = 10$$

$$2\times(4) \quad 2\alpha_2 + 4\alpha_3 = 6$$

$$2\alpha_3 = 4$$

$$\boxed{\alpha_3 = 2}$$

substitude $\boxed{\alpha_3 = 2}$ in eqn (4).

$$\alpha_2 + 2(2) = 3$$

$$\alpha_2 + 4 = 3$$

$$\boxed{\alpha_2 = -1}$$

substitude $\alpha_2 = -1$ and $\alpha_3 = 2$ in eqn (1)

$$\alpha_1 - 1 + 2 = 2$$

$$\alpha_1 + 1 = 2$$

$$\boxed{\alpha_1 = 1}$$

$$\boxed{\therefore a_n = (1)^n - (2)^n + 2(3)^n}$$

---

4) Find all the solution of the recurrence relation $S(n) - 3S(n-1) - 4S(n-2) = 4^n$.

The given relation can be rewritten as.

$$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$$

Characteristic equation:

$$r^2 - 3r - 4 = 0$$

$$(r+1)(r-4) = 0$$

$r + 1 = 0$, $r = -1$ | $r - 4 = 0$, $r = 4$

-4, -3, $\frac{-4}{r}$, $\frac{1}{r}$

$$\boxed{\therefore r = -1, 4}$$

$$\therefore a_n^h = \alpha_1(-1)^n + \alpha_2(4)^n$$

Particular solution:

$a_{n-1} = c(n-1)4^{n-1}$

Let $a_n = cn4^n$.

$$cn4^n - 3c(n-1)4^{n-1} - 4c(n-2)4^{n-2} = 4^n$$

$$\cancel{4^n}\left(cn - 3c(n-1)\frac{1}{4} - 4c(n-2)\frac{1}{16}\right) = \cancel{4^n}$$

$$cn - \frac{3}{4}cn + \frac{3c}{4} - \frac{1}{4}cn + \frac{c}{2} = 1$$

$$cn - cn + \frac{3c}{4} + \frac{c}{2} = 1$$

$$c\left(\frac{3}{4} + \frac{1}{2}\right) = 1$$

$$c\left(\frac{3+2}{4}\right) = 1$$

$$c\left(\frac{5}{4}\right) = 1$$

$$\boxed{c = \frac{4}{5}}$$

$$\therefore a_n^p = \frac{4}{5} n 4^n$$

The solution is...

$$a_n = a_n^h + a_n^p = \alpha_1(-1)^n + \alpha_2 4^n + \frac{4}{5} n 4^n$$

Result:

RHS of the form

i) $a_0 + a_1 n + \dots + a_r n^r$

then

$a_n = c_0 + c_1 n + \dots + c_r n^r$

ii) a^n is a root of characteristic equation

then $a_n = cna^n$.

iii) a^n is not the root of characteristic equation

then

$a^n = ca^n$.

iv) Constant means $a_n = c = a_{n-1} = a_{n-2} = \dots$

5) Find all the solution of the recurrence relation $G(K) - 7G(K-1) + 10G(K-2) = 8K+6$, $G(0) = 1$, $G(1) = 2$.

The given relation can be rewritten as,

$$a_n - 7a_{n-1} + 10a_{n-2} = 8n+6 \; ; \; a_0 = 1, a_1 = 2.$$

characteristic equation is.

$8n^2 + 7n + 6$

$$r^2 - 7r + 10 = 0$$

$$(r-5)(r-2) = 0$$

$$\therefore r = 2, 5$$

-7, 10, -5, -2, r, r

r, a, $an+d$

$$a_n^h = \alpha_1 2^n + \alpha_2 5^n$$

Particular solution:

$a_n = cn^2 + cn + d$

Let $a_n = cn + d$

$2n + 6$

$$cn + d - 7[c(n-1) + d] + 10[c(n-2) + d] = 8n+6$$

$$cn + d - 7(cn - c + d) + 10(cn - 2c + d) = 8n+6$$

$$cn + d - 7cn + 7c - 7d + 10cn - 20c + 10d = 8n+6$$

$$cn - 7cn + 10cn + d + 7c - 7d - 20c + 10d = 8n+6$$

$$4cn + 4d - 13c = 8n + 6.$$

Equating like coefficient on both sides,

$$4c = 8$$

$$\boxed{c = 2}$$

$$4d - 13c = 6$$

$$4d - 26 = 6$$

$$4d = 32$$

$$\boxed{d = 8}$$

$$\therefore a_n^P = 2n + 8$$

The solution is..

$$\therefore a_n = a_n^h + a_n^P = \alpha_1 2^n + \alpha_2 5^n + 2n + 8$$

$a_0 = 1$

$$a_0 = \alpha_1 (2)^0 + \alpha_2 (5)^0 + 2(0) + 8 = 1$$

$$\Rightarrow \alpha_1 + \alpha_2 = -7 \quad \text{—} (1)$$

$a_1 = 2$

$a_1 = 2\alpha_1 + 5\alpha_2 + 10 = 2$

$\Rightarrow 2\alpha_1 + 5\alpha_2 = -8 \quad -(2)$

$2\times(1) - (2)$

$2\times(1) \Rightarrow 2\alpha_1 + 2\alpha_2 = -14$

$(2) \Rightarrow 2\alpha_1 + 5\alpha_2 = -8$

$(-) \quad (-) \quad (+)$

$-3\alpha_2 = -6$

$\boxed{\alpha_2 = 2}$

Substitude α_2 value in equation (1)

$\alpha_1 + \alpha_2 = -7$

$\boxed{\alpha_1 = -9}$

$\boxed{\therefore a_n = -9(2)^n + 2(5)^n + 2n + 8}$

---

6) Find all the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

The given relation can be rewritten as,

$a_n - 5a_{n-1} - 6a_{n-2} = 7^n$

Characteristic equation is

$r^2 - 5r + 6 = 0$

$\therefore r = 2, 3$

$a_n^h = \alpha_1 2^n + \alpha_2 3^n$

Particular solution:

Let $a_n = c7^n$

$c7^n - 5c7^{n-1} + 6c7^{n-2} = 7^n$

$c - \frac{5c}{7} + \frac{6c}{49} = 1$

$\frac{49c - 35c + 6c}{49} = 1$

$$20c = 49$$

$$c = \frac{49}{20}$$

$$a_n^p = \frac{49}{20} \cdot 7^n = \frac{7^{n+2}}{20}$$

The solution is...

$$a_n = a_n^h + a_n^p$$

$$a_n = \alpha_1 2^n + \alpha_2 3^n + \frac{7^{n+2}}{20}$$

---

7) Find all the solution of recurrence relation $S(k) - 5S(k-1) + 6S(k-2) = 2$, $S(0) = 1$, $S(1) = -1$.

The given relation can be rewritten as,

$$a_n - 5a_{n-1} + 6a_{n-2} = 2, \quad a_0 = 1, \quad a_1 = -1$$

Characteristic equation is

$$r^2 - 5r + 6 = 0.$$

$$r = 2, 3$$

$$a_n^h = \alpha_1 2^n + \alpha_2 3^n$$

Particular equation:

Let $a_n = c$.

$$c - 5c + 6c = 2$$

$$2c = 2$$

$$c = 1$$

$$\therefore a_n^p = 1$$

The solution is...

$$a_n = a_n^h + a_n^p$$

$$a_n = \alpha_1 2^n + \alpha_2 3^n + 1$$

$a_0 = 1$

$$a_0 = \alpha_1 (2)^0 + \alpha_2 (3)^0 + 1 = 1$$

$$\Rightarrow \alpha_1 + \alpha_2 = 0 \quad -(1)$$

$\alpha_1 = -1$

$$a_1 = 2\alpha_1 + 3\alpha_2 + 1 = -1$$

$$\Rightarrow 2\alpha_1 + 3\alpha_2 = -2 \quad —(2)$$

$2\times(1) - (2)$

$$2\times(1) \Rightarrow 2\alpha_1 + 2\alpha_2 = 0$$

$$(-)$$

$$(2) \Rightarrow 2\alpha_1 + 3\alpha_2 = -2$$

$$(-) \quad (-) \quad (+)$$

$$-\alpha_2 = +2$$

$$\boxed{\alpha_2 = -2}$$

substitude α_2 value in (1).

$$\alpha_1 - 2 = 0$$

$$\boxed{\alpha_1 = +2}$$

$$\boxed{a_n = 2(2)^n - 2(3)^n + 1}$$

Generating function

The generating function for the sequence 's' with terms $a_0, a_1, \ldots, a_n$ of real numbers is the infinite sum.

$$G(x) = a_0 + a_1x + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n.$$

Note:

$$1 + x + x^2 + x^3 + \ldots = (1-x)^{-1}$$

$$1 - x + x^2 - x^3 + \ldots = (1+x)^{-1}$$

$$1 + 2x + 3x^2 + \ldots = (1-x)^{-2}$$

$$1 - 2x + 3x^2 - \ldots = (1+x)^{-2}$$

1) Solve the recurrence relation $a_n = 3a_{n-1} + 1, n \geq 1, a_0 = 1$ using the generating function.

Given:

$$a_n - 3a_{n-1} = 1 \quad , n \geq 1, a_0 = 1$$

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 1 \cdot x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^{n-1+1} = \sum_{n=1}^{\infty} x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n-1}^{\infty} x^n$$

$$G(x) - a_n - 3x\,G(x) = x + x^2 + x^3 + \dots$$

$$G(x)(1-3x) - 1 = x(1+x+x^2+\dots) = x(1-x)^{-1}$$

$$(\because (1+x+x^2+\dots) = (1-x)^{-1})$$

$$G(x)(1-3x) - 1 = \frac{x}{1-x}$$

$$G(x)(1-3x) = \frac{x}{1-x} + 1 = \frac{x+1-x}{1-x} = \frac{1}{1-x}$$

$$\therefore G(x) = \frac{1}{(1-x)(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-3x)}$$

$$1 = A(1-3x) + B(1-x)$$

put $x = 1$

$1 = -2A$

$\boxed{A = -1/2}$

put $x = 0$

$1 = A + B$

$1 = -1/2 + B$

$1 + 1/2 = B$

$\boxed{B = 3/2}$

$$\therefore G(x) = \frac{-1/2}{1-x} + \frac{3/2}{1-3x}$$

$$= \frac{-1}{2}(1-x)^{-1} + \frac{3}{2}(1-3x)^{-1}$$

$$= \frac{-1}{2}(1+x+x^2+\dots) + \frac{3}{2}(1+3x+(3x)^2+\dots)$$

$$= \frac{-1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} (3x)^n$$

$$= \frac{-1}{2}\sum_{n=0}^{\infty} 1^n x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$\boxed{\therefore a_n = \frac{-1}{2} 1^n + \frac{3}{2} 3^n}$$

---

2) Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ with $a_0 = 1$ using generating function.

Given: $a_n = 8a_{n-1} + 10^{n-1}$, $a_0 = 1$

$$a_n - 8a_{n-1} = 10^{n-1}, \quad a_0 = 1$$

$$\sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 10^{n-1} x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 8x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \frac{1}{10}\sum_{n=1}^{\infty} 10^n x^n$$

$$\cancel{\sum_{n=1}^{\infty} a_n x^n +}$$

$$G(x) - a_0 - 8x\, G(x) = \frac{1}{10}\sum_{n=1}^{\infty} (10x)^n$$

$$G(x)(1-8x) - 1 = \frac{1}{10}\left(10x + (10x)^2 + \cdots\right)$$

$$= \frac{1}{10}(10x)\left(1 + 10x + (10x)^2 + \cdots\right.$$

$$G(x)(1-8x) - 1 = x(1-10x)^{-1}$$

$$G(x)(1-8x) = \frac{x}{1-10x} + 1 = \frac{x+1-10x}{1-10x}$$

$$G(x)(1-8x) = \frac{1-9x}{1-10x}$$

$$G(x) = \frac{1-9x}{(1-10x)(1-8x)} = \frac{A}{(1-10x)} + \frac{\ }{(1-}$$

$1-9x = A(1-8x)+B(1-10x)$

put $x=0$ $\qquad$ put $x=1$

$1 = A+B \quad -(1)$ $\qquad$ $-8 = -7A-9B \quad -(2)$

$7\times(1) \Rightarrow 7A+7B = 7$

$(2) \Rightarrow -7A-9B = -8 \quad (+)$

$-2B = -1$

$\boxed{B = 1/2}$ $\qquad$ $\boxed{A = 1/2}$

$\therefore G(x) = \dfrac{1/2}{1-10x} + \dfrac{1/2}{1-8x}$

$= \frac{1}{2}(1-10x)^{-1} + \frac{1}{2}(1-8x)^{-1}$

$= \frac{1}{2}(1+10x+(10x)^2+\dots) + \frac{1}{2}(1+8x+(8x)^2+\dots)$

$= \frac{1}{2}\sum_{n=0}^{\infty}(10x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(8x)^n$

$\therefore a_n = \frac{1}{2}(10)^n + \frac{1}{2}(8)^n$

$\boxed{\therefore a_n = \frac{1}{2}(10^n+8^n)}$

---

3) Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $a_0 = 2$, $a_1 = 1$ using generating function.

Given: $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $a_0 = 2$, $a_1 = 1$

$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$

$\sum_{n=0}^{\infty} a_{n+2}x^{n+2-2} - 2\sum_{n=0}^{\infty} a_{n+1}x^{n+1-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(2x)^n$

$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - \frac{2}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(2x)^n$

$= (1+2x+(2x)^2+\dots$

$$\frac{1}{x^2}\left(G(x)-a_0-a_1x\right)-\frac{2}{x}\left(G(x)-a_0\right)+G(x)=(1-2x)^{-1}$$

$$\frac{1}{x^2}G(x)-\frac{2}{x^2}-\frac{1}{x}-\frac{2}{x}G(x)+\frac{4}{x}+G(x)=\frac{1}{1-2x}$$

$$\frac{G(x)-2-x-2xG(x)+4x+x^2G(x)}{x^2}=\frac{1}{1-2x}$$

$$G(x)(1-2x+x^2)-2+3x=\frac{x^2}{1-2x}$$

$$G(x)(1-2x+x^2)=\frac{x^2}{1-2x}+2-3x$$

$$=\frac{x^2+(2-3x)(1-2x)}{(1-2x)}$$

$$=\frac{x^2+2-4x-3x+6x^2}{1-2x}$$

$$=\frac{7x^2-7x+2}{1-2x}$$

$$G(x)=\boxed{\frac{7x^2-7x+2}{(1-2x)(1-2x+x^2)}}$$

$$=\frac{7x^2-7x+2}{(1-2x)(x-1)^2}$$

$$G(x)=\frac{A}{1-2x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

$$7x^2-7x+2=A(x-1)^2+B(1-2x)(x-1)+C(1-2x)$$

put $x=1$: $2=-C$, $\boxed{C=-2}$

put $x=0$: $2=A-B+C$, $\boxed{A-B=4}$ —(2)

put $x=-1$: $7+7+2=4A-6B+3C$, $\boxed{22=4A-6B}$ —(3)

$$(1)\times 6 \Rightarrow 6A - 6B = 24$$

$$(2) \Rightarrow \underset{(-)}{4A} \underset{(+)}{-6B} = \underset{(-)}{22} \quad (-)$$

$$2A = 2$$

$$A = 1 \qquad B = -3$$

$$\therefore G(x) = \frac{1}{(1-2x)} - \frac{3}{(x-1)} - \frac{2}{(x-1)^2}$$

$$= \frac{1}{1-2x} + \frac{3}{1-x} - \frac{2}{(1-x)^2}$$

$$= (1-2x)^{-1} + 3(1-x)^{-1} - 2(1-x)^{-2}$$

$$= (1+2x+(2x)^2+\dots) + 3(1+x+x^2+\dots) - 2(1+2x+3x^2+\dots)$$

$$= \sum_{n=0}^{\infty} (2x)^n + 3\sum_{n=0}^{\infty} x^n - 2\sum_{n=0}^{\infty} (n+1)x^n$$

$$= \sum_{n=0}^{\infty} 2^n x^n + 3\sum_{n=0}^{\infty} 1^n x^n - 2\sum_{n=0}^{\infty} (n+1)x^n$$

$$a_n = 2^n + 3 - 2(n+1)$$

$$= 2^n + 3 - 2n - 2$$

$$\boxed{\therefore a_n = 2^n - 2n + 1}$$

The principle of Inclusion - Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4|$$

$+ |A_1 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

For example:....

Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f\}$

$|A| = 3$, $|B| = 5$, $|A \cup B| = 6$, $|A \cap B| = 2$

$A \cup B = \{a, b, c, d, e, f\}$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 3 + 5 - 2$

$= 8 - 2 = 6$

---

1) A total of 1232 students have taken a course in tamil, 879 have taken a course in Telugu and 114 have taken a course in Hindi. Further 10 have taken a course in both tamil and telugu, 23 have taken a course in tamil and hindi and 14 have taken a course in telugu and hindi. If 2092 students have taken atleast one of the tamil, telugu and hindi, how many students have taken a course in all three languages?

Let 'A' denote set of students have taken a course in tamil.

'B' denote set of students have taken a course in telugu.

'C' denote set of students have taken a course in hindi

$|A| = 1232$, $|B| = 879$, $|C| = 114$

$|A \cap B| = 103$, $|A \cap C| = 23$, $|B \cap C| = 14$

$|A \cup B \cup C| = 2092$

$|A \cap B \cap C| = ?$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$2092 = 1232 + 879 + 114 - 103 - 14 - 23 + |A \cap B \cap C|$

$2092 = 2085 + |A \cap B \cap C|$

$2092 - 2085 = |A \cap B \cap C|$

$\boxed{|A \cap B \cap C| = 7}$

$\therefore$ 7 students have taken a course in all three languages.

---

2) Find the number of integers between 1 to 250 that ~~all~~ are not divisible by any of the integers 2, 3, 5 and 7.

Let 'A' denote the integer from 1 to 250 that are divisible by 2.

'B' denote the integer from 1 to 250 that are divisible by 3.

'C' denote the integer from 1 to 250 that are divisible by 5.

'D' denote the integer from 1 to 250 that are divisible by 7.

$|A| = \left[\frac{250}{2}\right] = 125$

$|B| = \left[\frac{250}{3}\right] = 83$

$|C| = \left[\frac{250}{5}\right] = 50$

$|D| = \left[\frac{250}{7}\right] = 35$

$|A \cap B| = \left[\frac{250}{2\times3}\right] = 41$

$|A \cap C| = \left[\frac{250}{2\times5}\right] = 25$

$|B \cap D| = \left[\frac{250}{3\times7}\right] = 11$

$|C \cap D| = \left[\frac{250}{5\times7}\right] = 7$

$|A \cap D| = \left[\frac{250}{2\times7}\right] = 17$

$|B \cap C| = \left[\frac{250}{3\times5}\right] = 16$

$$|A\cap B\cap C| = \left\lfloor \frac{250}{2\times3\times5} \right\rfloor = 8$$

$$|A\cap B\cap B| = \left\lfloor \frac{250}{2\times3\times7} \right\rfloor = 5$$

$$|A\cap C\cap D| = \left\lfloor \frac{250}{2\times5\times7} \right\rfloor = 3$$

$$|B\cap C\cap D| = \left\lfloor \frac{250}{3\times5\times7} \right\rfloor = 2$$

$$|A\cap B\cap C\cap D| = \left\lfloor \frac{250}{2\times3\times5\times7} \right\rfloor = 1$$

$$|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D|-|A\cap B|-|A\cap C|-|A\cap D|$$
$$-|B\cap C|-|B\cap D|-|C\cap D|+|A\cap B\cap C|+|A\cap C\cap D|$$
$$+|A\cap B\cap D|+|B\cap C\cap D|-|A\cap B\cap C\cap D|$$
$$=125+83+50+35-41-25-17-16-11-7$$
$$+8+3+2+5-1$$
$$=193$$

$\therefore$ The number of integers from 1 to 250 that are not divisible by 2,3,5,7 $= 250-193 = \underline{\underline{57}}$

---

3) How many integers between 1 to 100 that are i) Not divisible by 7, 11 and 13

ii) Divisible by 7 but not by 11

Let 'A' denote the set of numbers that are divisible 7

Let 'B' denote the set of numbers that are divisible 11

Let 'C' denote the set of numbers that are divisible 13

i) $|A| = \left|\frac{100}{7}\right| = 14$ $|B| = \left|\frac{100}{11}\right| = 9$ $|C| = \left|\frac{100}{13}\right| = 7$

$|A \cap B| = \left|\frac{100}{7 \times 11}\right| = 1$ $|A \cap C| = \left|\frac{100}{7 \times 13}\right| = 1$ $|B \cap C| = \left|\frac{100}{11 \times 13}\right| = 0$

$|A \cap B \cap C| = \left|\frac{100}{7 \times 11 \times 13}\right| = 0$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 14 + 9 + 7 - 1 - 1 - 0 + 0$$
$$= 28$$

$\therefore$ The number of integers from 1 to 100 that are not divisible by 7, 11, 13 $= 100 - 28 = \underline{\underline{72}}$

ii) The no. of integers from 1 to 100 that are divisible by 7 but not by 11 $= |A| - |A \cap B|$

$$= 14 - 1 = \underline{\underline{13}}$$

---

4) A survey among ~~[illegible]~~ 100 students shows that of the 3 icecream flavours vennila, chocolate and strawberry, 50 students like vennila, 43 like chocolate, 28 like strawberry, 13 like vennila and chocolate, 11 like chocolate and strawberry, 12 like strawberry and vennila and 5 like all of them.

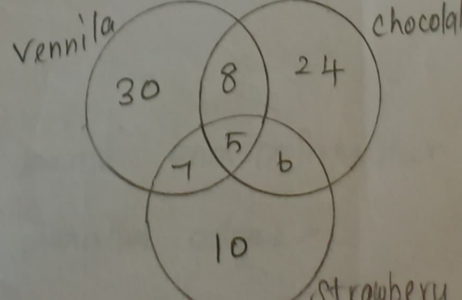
Find the no. of student survey who like each of the following.

i) Chocolate but not strawberry

ii) Chocolate and strawberry but not vennila

iii) Vennila or Chocolate but not strawberry.

Diagram:



Let 'v' denote the set of students who like Vennila.

Let 'c' denotes the set of students who like chocolate

Let 's' denotes the set of students who like strawberry

Given

$|v| = 50$, $|c| = 43$, $|s| = 28$, $|v \cap c| = 13$, $|c \cap s| = 11$,

$|s \cap v| = 12$, $|v \cap c \cap s| = 5$

i) The no. of students who like Chocolate but not strawbery $= |c - s| = 24 + 8 = 3$

ii) The no. of students who like chocolate and strawberry but not vennila $= |(c \cap s) - v| = 6$

iii) The no. of students who like vennila or chocolate but not strawberry $= |(v \cup c) - s|$

$= 30 + 8 + 24 = 62$

Unit-3

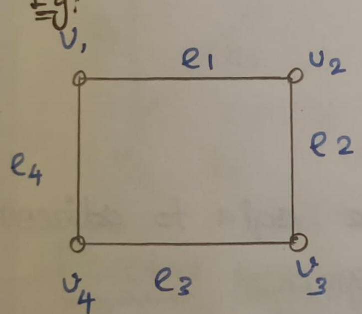
GRAPHS

Graph:

A graph $G = (V, E, \phi)$ consists of a non empty set $V = \{v_1, v_2, \ldots\}$ called the set of nodes (Points, Vertices) of the graph $E = \{e_1, e_2, \ldots\}$ is said to be the set of edges of the graph, and ϕ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V.

The vertices are represented by points and each edge is represented by a line diagrammatically.

Eg:

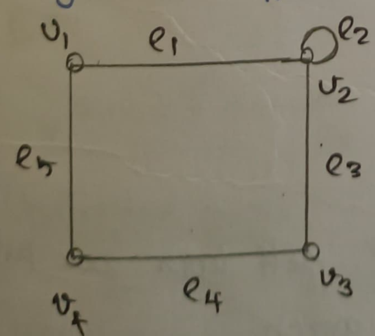


v_1, v_2, v_3, v_4 are called vertices

e_1, e_2, e_3, e_4 are called edges.

Self loop:

If there is an edge from v_1 to v_2 then that edge is called self loop or simply loop.

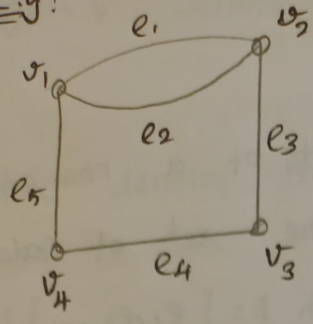


Here e_2 is a loop.

Parallel edges:

If two edges have same end points then the edges are called parallel edges.

E.g:

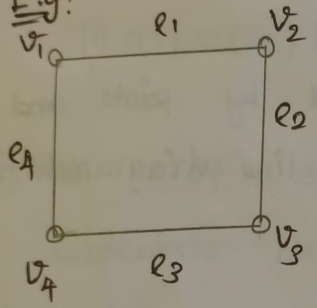


Here e_1 and e_2 are parallel edges.

Adjacent edges:

Two edges are said to be adjacent if they are incident on a common vertex.

E.g:

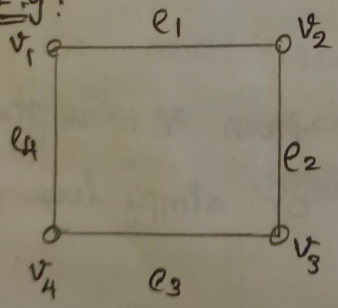


Here e_1 and e_2, e_2 and e_3, e_3 and e e_1 and e_4 are adjacent edges

Adjacent vertices:

Two vertices v_i and v_j are said to adjacent if v_i, v_j is an edge of the graph.

E.g:

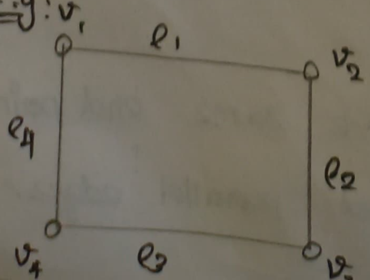


Here v_1 and v_2, v_2 and v_3, v_3 and v v_1 and v_4 are adjacent vertices.

Simple graph:

A graph which has neither self loops nor parallel edges is called a simple graph.

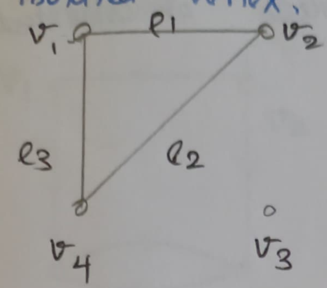
E.g:



Isolated vertex:

A vertex having no edge incident on it is called an isolated vertex.

E.g:

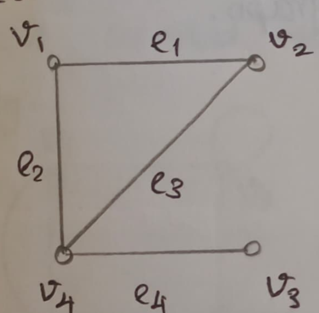


Here v_3 is called an isolated vertex.

Pendent vertex:

If the degree of any vertex is one, then that vertex is called pendent vertex.

E.g:

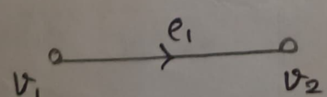


Here v_3 is an pendent vertex.

Directed edges:

In a graph $G=(V,E)$, on edge which is associated with an ordered pair of $V \times V$ is called a directed edge of G.

E.g:

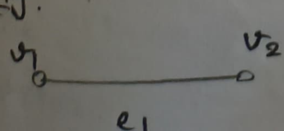


Here, e_1 is directed edge

Undirected edge:

If an edge which is associated with an unorded pair of nodes is called an undirected edge.

E.g:

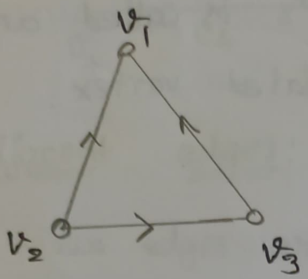


Here e_1 is undirected edge.

Digraph:

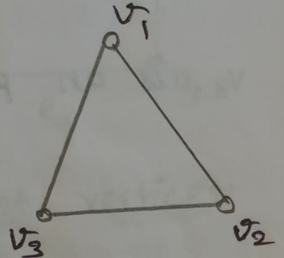
A graph in which every edge is directed edge is called a digraph or directed graph.

E.g:



Undirected graph:

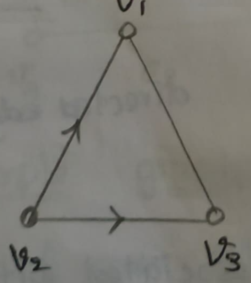
A graph in which every edge is undirected is called an undirected graph.



Mixed graph:

If some edges are directed and some are undirected in a graph, the graph is called mixed graph.

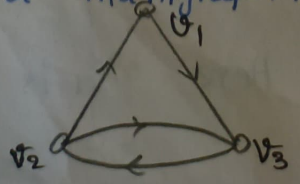
E.g:



Multi graph:

A graph which contains some parallel edges is called a multigraph.

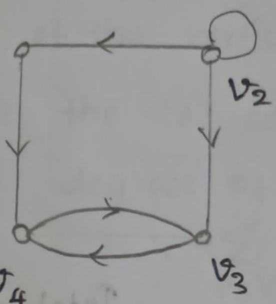
E.g:



Pseudograph:

A graph in which loops and parallel edges are allowed is called a pseudograph.

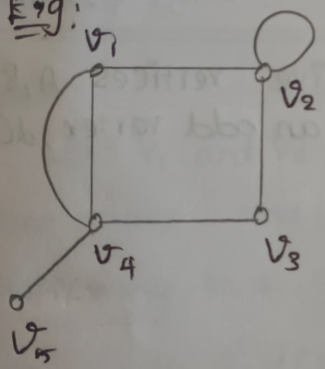
E.g:



Degree of a vertex:

The no. of edges incident at the vertex v_i is called the degree of the vertex with self loops counted twice and it is denoted by $d(v_i)$.

E.g:



$d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 3$

$d(v_4) = 4$, $d(v_5) = 1$, $d(v_6) = 0$.

In-degree and out-degree of a directed graph:

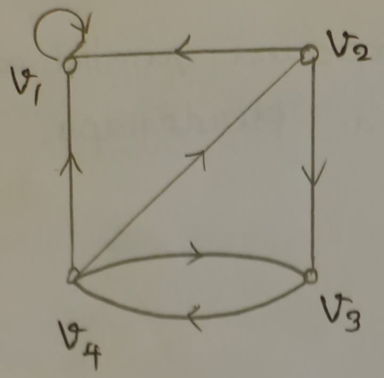
In a directed graph, the in-dagree of a vertex V, denoted by $deg^-(V)$ and defined by the no. of edges with V as their terminal vertex.

The out-degree of V, denoted by $deg^+(V)$, is the no. of edges with V as their initial vertex.

Note:

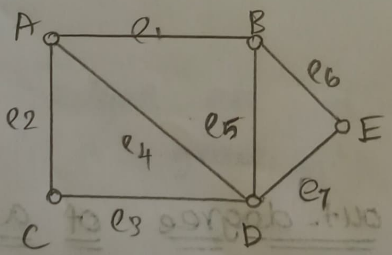
A loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

E.g:



In-degree	Out-degree	Total degree
$deg^-(v_1)=3$	$deg^+(v_1)=1$	$deg(v_1)=4$
$deg^-(v_2)=1$	$deg^+(v_2)=2$	$deg(v_2)=3$
$deg^-(v_3)=2$	$deg^+(v_3)=1$	$deg(v_3)=3$
$deg^-(v_4)=1$	$deg^+(v_4)=3$	$deg(v_4)=4$

1) Draw the graph with five vertices A, B, C, D, E such that degree (A) = 3, B is an odd vertex, d(C) = 2, D and E are adjacent.



Theorem 1: The Handshaking theorem

Let $G=(V,E)$ be an undirected graph with e edges. Then

$$\sum_{v \in V} deg(v) = 2e.$$

that is The sum of degrees of all the vertices of an undirected graph is twice the no. of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the 'e' edges contribute $2e$ to the sum of the degrees of vertices.

$$\therefore \sum_{v \in V} \deg(v) = 2e.$$

Hence the proof.

Theorem 2:

In a undirected graph, the no. of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph $G = (V, E)$.

$$\therefore \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

By handshaking theorem, we have

$$2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j) \quad -(1)$$

Since each $d(v_i)$ is even, $\sum_{v_i \in V_1} d(v_i)$ is even.

As LHS of equation (1) is even and the first expression on the RHS of equation (1) is even, we have the 2^{nd} expression on the RHS must be even.

$$\therefore \sum_{v_j \in V_2} d(v_j) \text{ is even.}$$

Since each $d(v_j)$ is odd, the no.of terms contained in $\sum_{v_j \in V_2} d(v_j)$ must be even.

Hence the no.of vertices of odd degree is even.

Hence the proof.

Theorem 3:

The maximum no.of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof:

We prove this theorem by the principle of mathematical induction. For n=1, a graph with one vertex has no edges.

$\therefore$ The result is true for n=1.

For n=2, a graph with 2 vertices may have atmost one edge.

$$\therefore \frac{2(2-1)}{2} = 1.$$

$\therefore$ The result is true for n=2.

Assume that the result is true for n=k. that is a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges

When n = k+1, let G be a graph having n vertices and G' be the graph obtained from G by deleting one vertex say $v \in V(G)$.

Since G' has k vertices, then by the hypothesis G' has atmost $\frac{k(k-1)}{2}$ edges. Now add the vertex v to G' such that v may be adjacent to all the k vertices of G'.

$\therefore$ The total no.of edges in G are

$$\frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k^2 + k}{2} = \frac{k(k+1)}{2} = \frac{(k+1)(k+1-1)}{2}$$

$\therefore$ The result is true for $n = k+1$.

Hence the maximum no.of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Hence the proof.

1) How many edges are there in a graph with 10 vertices each of degree 6?

Given: $n = 10$

We know that, $\sum_{v \in V} d(v) = 2e$

$$10 \times 6 = 2e$$
$$60 = 2e$$
$$e = \frac{60}{2} = 30.$$

2) Can a simple graph exists with 15 vertices each of degree 5.

Given: $n = 15$

We know that $\sum_{v \in V} d(v) = 2e$

$$15 \times 5 = 2e$$
$$75 = 2e$$
$$e = \frac{75}{2}$$

$\therefore$ The simple graph doesn't exists.

3) For the following degree sequence 4, 4, 4, 3, 2, find if there exist a graph or not?

Given degree sequence 4, 4, 4, 3, 2.

We known that $\sum_{v \in V} d(v) = 2e$

$$4+4+4+3+2 = 2e$$
$$17 = 2e$$
$$e = \frac{17}{2}$$

$\therefore$ The graph doesn't exists.

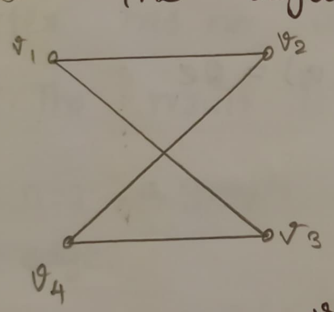
Adjacency Matrix

$$A = [a_{ij}] = \begin{cases} 1, & \text{if there exist an edge between } v_i \,\&\, v_j \\ 0, & \text{otherwise} \end{cases}$$

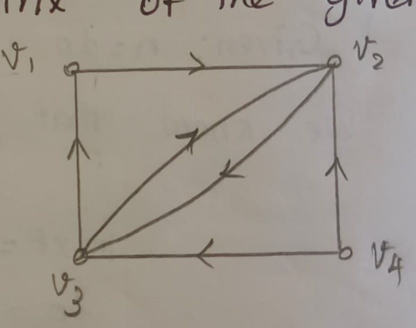
---

1) Find the adjacency matrix of the given below

(i)



(ii)



(i) $A = [a_{ij}] = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 1 & 0 \end{array}$

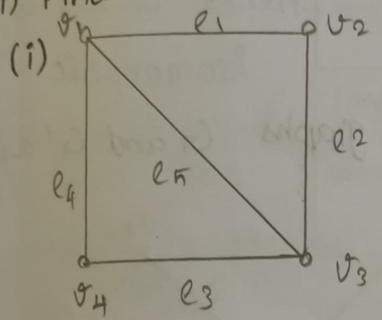
(ii) $A = [a_{ij}] = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 0 \\ v_4 & 0 & 1 & 1 & 0 \end{array}$

Incidence Matrix:

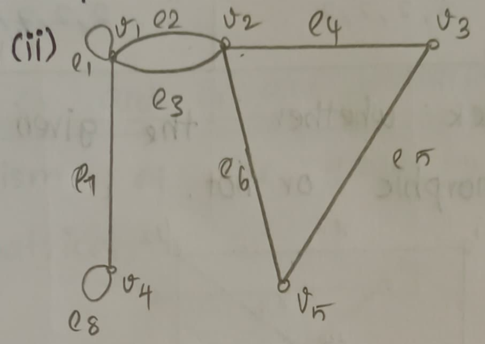
$$B = [b_{ij}] = \begin{cases} 1, & \text{when edge } e_j \text{ incident on } v_i \\ 0, & \text{otherwise} \end{cases}$$

1) Find incidence matrix of (i)

(i)



(ii)



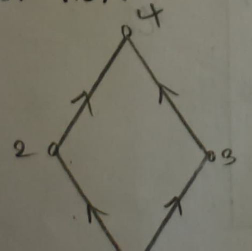
(i) $B = [b_{ij}] =$

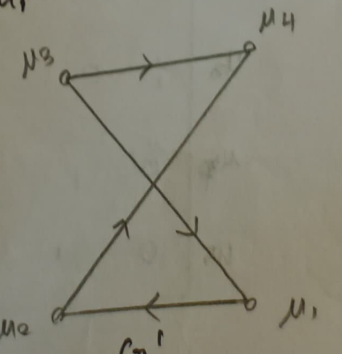
$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 1 & 0 & 0 & 1 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 1 & 0 \end{array}$$

(ii) $B = [b_{ij}] =$

$$\begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ v_2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

2) Check the given 2 graphs G and G' are isomorphic or not.





G	G'
(i) No. of vertices $n=2$	(i) No. of vertices $n=9$
(ii) No. of edges $e=4$	(ii) No. of edges $e=4$
(iii) Degree sequence 2, 2, 2, 2	(iii) Degree sequence 2, 2, 2, 2

Mapping:

$1 \to u_3$

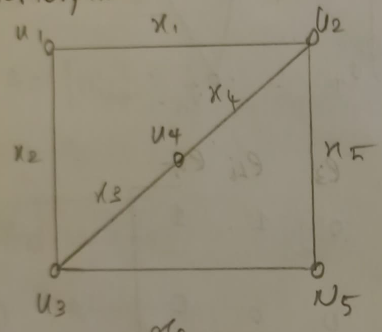
$2 \to u_1$

$3 \to u_2$

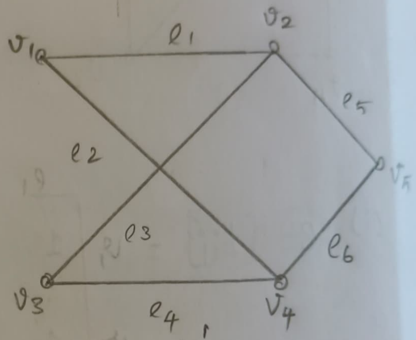
$4 \to u_4$

∴ Hence G and G' are isomorphic

3) Check whether the given 2 graphs G and G' are isomorphic or not.



G



G'

G	G'
(i) No. of vertices $n=5$	(i) No. of vertices $n=5$
(ii) No. of edges $e=6$	(ii) No. of edges $e=6$
(iii) Degree sequence 2, 3, 3, 2, 2	(iii) Degree Sequence 2, 3, 2, 3, 2

Mapping:

$u_1 \to v_1$

$u_2 \to v_2$

$u_3 \to v_4$

$u_4 \to v_5$

$u_5 \to v_3$

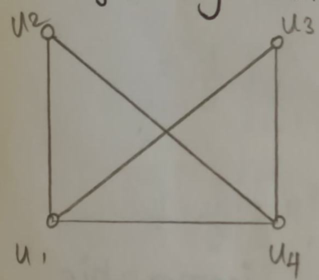
Checking process

$$A = [a_{ij}] = \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 1 & 0 & 0 \\ u_2 & 1 & 0 & 0 & 1 & 1 \\ u_3 & 1 & 0 & 0 & 1 & 1 \\ u_4 & 0 & 1 & 1 & 0 & 0 \\ u_5 & 0 & 1 & 1 & 0 & 0 \end{array}$$

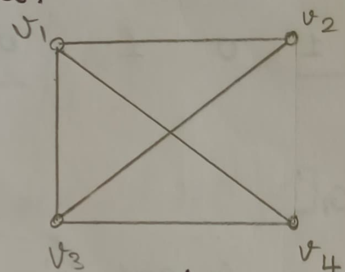
$$B=[b_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_4 \\ v_5 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_4 & v_5 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore$ Hence G and G' are isomorphi.

4) Test the isomorphism of the graph by considering their adjacency matrices.



G



G'

Adjacency matrices

$$G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad G' = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$= G'$

Hence G and G' are isomorphic.

---

5) Prove that the simple graph with n vertices & k component cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges,

Let $n_1, n_2, \dots n_k$ be the no.of vertices of each k component respectively.

$$\sum_{i=1}^{k} n_i = n_1 + n_2 + \dots n_k = n = |V(G)|$$

$$\sum_{i=1}^{n} (n_i - 1) = n_1 - 1 + n_2 - 1 + \dots n_k - 1$$

$$= n_1 + n_2 + \dots n_k - k$$

$$\sum_{i=1}^{n} (n_i - 1) = n - k$$

squaring on both side we get

$$\left(\sum_{i=1}^{k} (n_i - 1)\right)^2 = (n-k)^2$$

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \dots (n_k - 1)^2 \le n^2 + k^2 - 2nk$$

$$n^2 + 1 - 2n_1 + n_2^2 + 1 - 2n_2 + \dots + n_k^2 + 1 - 2nk \le n^2 + k^2 -$$

$n_1^2 + n_2^2 + \dots + n_k^2 - 2(n_1 + n_2 + \dots n_k) \le n^2 + k^2 - 2nk$

$$\sum_{i=1}^{k} n_i^2 + k - 2n \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk - k + 2n$$

Since G is simple the mean no. of edges in each component is $\frac{n_i(n_i-1)}{2}$ edges.

Maximum no. of edges in edges in $G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i^2 - n_i)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[n^2 + k^2 - 2nk - k + 2n - n\right]$$

$$= \frac{1}{2}\left[(n-k)^2 - k + n\right] = \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Mean no. of edges in $G \le \frac{(n-k)(n-k+1)}{2}$

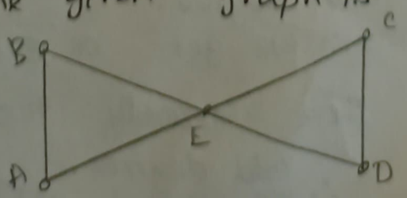
Result:

(1) A connected graph is Euler graph iff each of its vertices is of even degree.

(2) A connected graph has an Euler path but not an Euler circuit iff has exactly two vertices of odd degree.

(3) A graph with a vertex of degree 1 cannot have a Hamiltonion cycle.

1) Check the given graph is Euler or not.



deg(A) = 2

deg(B) = 2

deg(C) = 2

deg(D) = 2

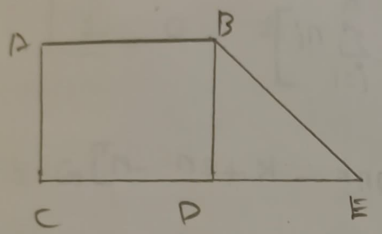
deg(E) = 4

Here all the vertices are even degree.

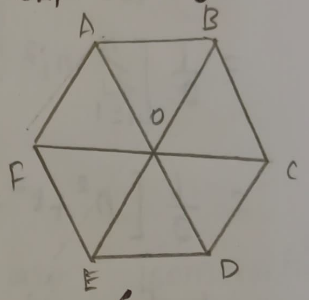
∴ The given graph is Euler.

---

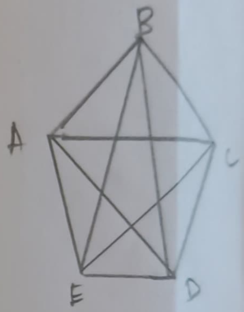
2) Find an Euler path or an Euler circuit, if is exit in each of the following 3 graphs. If it doesn't exit explain why?



G_1



G_2



G_3

G_1:

deg(A) = 3

deg(B) = 3

deg(C) = 2

deg(D) = 4

deg(E) = 2

Here we get exactly 2 vertices of odd degree

∴ We get a Euler path

Here all the vertices are not have an even degree

∴ we cannot get a Euler circuit.

G_2:

deg(A) = 3 , deg(O) = 6

deg(B) = 3

deg(C) = 3

deg(D) = 3

deg(E) = 3

deg(F) = 3

Here, neither all vertices have even degree nor exactly 2 vertices of odd degree

∴ so, we cannot get both euler path and Euler circuit.

G_3:

deg(A) = 4

deg(B) = 4

deg(C) = 4

deg(D) = 4

deg(E) = 4

Here all the vertices have an even degree.

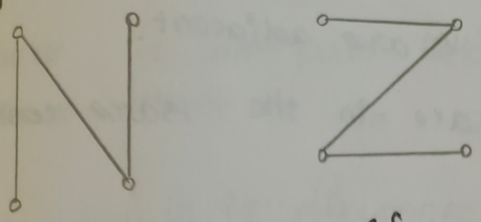
∴ We get a Euler circuit.

Here exactly two vertices cannot have odd degree.

∴ we cannot get Euler path.

A self complementry graph is a graph which is isomorphic to its complement.

E.g



G $\qquad$ G^c

1) Show that if a graph with n vertices is self-complementry then $n \equiv 0$ (or) $1 \pmod 4$

Proof :

Let $G=(V,E)$ be any self-complementary graph with n vertices.

We know that $G \cong G^c$ and $|E(G)| = |E(G^c)|$

$|E(G)| + |E(G^c)|$ = Total no. of ~~of~~ edges in k_n

$$= \frac{n(n-1)}{2}$$

$$\therefore 2|E(G)| = \frac{n(n-1)}{2}$$

$$|E(G)| = \frac{n(n-1)}{4}$$

$\therefore$ Either $n=4k$ (or) $n-1=4k$

$\therefore$ $n=4k$ (or) $n=4k+1$

Hence $n \equiv 0$ (or) $1 \pmod 4$

Theorem:

2) The complement of a disconnected graph is connected.

Proof :

Let G be a disconnected graph.

Let $G_1, G_2, \ldots\ldots G_k$ be the connected components of G

Case (i):

Let $u \in G_i$, $v \in G_j$, $i \neq j$

u and v are not adjacent in G.

So, in G^c, u and v are adjacent.

$\therefore$ In G^c, u & v are in the same connected component.

Case (ii):

Consider two vertices in the same connected component G_i.

Let $w \in G_j$, $i \neq j$

Both u and v are adjacent with w in G^c.

So, in G^c, there is a path uwv.

There u and v are connected in G^c

Hence the proof.

3) Prove that a graph is disconnected if and only if its vertices set can be particiant into two nonempty disjoint V_1 and V_2 such that there exists no edge in G whose one end vertices is in sustant V_1 and the another end sustant V_2

Let G be a disconnected graph.

Consider a vertex u in G.

Let V_1 be the set of all vertices reachable from u.

Since G is disconnected, V_1 does not contain all vertices of G.

Let V_2 be the remaining vertices of G.

$\therefore$ No vertex in V_1 is joined to any vertex in V_2 by edge.

Conversely, assume that G is a graph whose vertex can be partitioned into 2 nonempty disjoint subsets V_1 & V_2 such that no edge of G has one end in V_1 & the other in V_2.

Let u and w be any 2 vertices in G. such that $u \in V_1$ and $w \in V_2$.

There is no path between vertices u and w, since there is no edge joining them.

$\therefore$ G is disconnected.

---

4) Prove that maximum no. of edges in a bipartite graph with n vertices is $\frac{n^2}{4}$.

Let G be the complete bipartite graph with n vertices.

Maximum no. of edges in bipartite graph $\leq$ Maximum no. of edges in complete bipartite graph

V(G) can be partitened into 2 vertex set V_1 & V_2 such that,

$$|V_1| + |V_2| = |V(G)|$$

$$|V_1| + |V_2| = n$$

Edges will be maximum if either $|V_1| = \left[\frac{n}{2}\right]$ or $|V_2| = \left[\frac{n}{2}\right]$

$$\therefore \text{Maximum no. of edges} \leq \left[\frac{n}{2}\right] \times \left[\frac{n}{2}\right]$$

$$= \frac{n^2}{4}$$

---

5) Let G be a simple undirected graph with adjacency matrix A with respect to the ordinary $v_1, v_2, \dots, v_n$. Prove that the no. of difference walks of length r from v_i to v_j equals the $(i,j)^{th}$ entry of A^r, where r is a positive integer,

Proof:

Let $P(r)$: The no.of different walks of length r from v_i to v_j equals the $(i,j)^{th}$ entry of A^r.

Step 1: $r=1$

$$A = [a_{ij}]_{n\times n} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Here $(i,j)^{th}$ entry of A is the no.of walk of length 1 from v_i to v_j.

$\therefore$ The result is true for $r=1$

Step 2:

Assume that the result is true for $n=k$.

that is, $A^k = [a_{ij}]^k$

that is $(i,j)^{th}$ entry of A^k is the no.of walks of length k from v_i to v_j.

Claim: $r=k+1$ is true.

$$A^{k+1} = A^k . A = [a_{ij}^k] . [a_{ij}]$$

$$\begin{pmatrix} a_{11}^k & a_{12}^k & \cdots & a_{1n}^k \\ a_{21}^k & a_{22}^k & \cdots & a_{2n}^k \\ \vdots & & & \\ a_{i1}^k & a_{i2}^k & \cdots & a_{in}^k \\ \vdots & & & \\ a_{n1}^k & a_{n2}^k & \cdots & a_{nn}^k \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

$(i,j)^{th}$ entry of A^{k+1} $= a_{i1}^k a_{1j} + a_{i2}^k a_{2j} + \cdots + a_{in}^k a_{nj}$

$=$ (No.of walks of length k from v_i to v_1) $\times a_{1j}$

+ (No. of walks of length k from v_i to v_2) $\times a_{2j}$

+ . . . . . .

+ (No. of walks of length k from v_i to v_n) $\times a_{nj}$

From the above we got $(i,j)^{th}$ entry of A^{k+1} is the no. of walks of length $k+1$ from v_i to v_j.

This completes the proof.

---

6) A connected graph G is an Euler graph iff all vertices of G are of even degree.

Proof: Assumes that G is an Euler graph. Then G has an Euler cycle, say

$$C: \{u, v_1, v_2, \ldots, v_n, u\}$$

Clearly, if v is an internal vertex. Then

$$d(v) = 2 \times \begin{cases} \text{The no. of times } v \text{ occurs in} \\ \text{the Euler circuit} \end{cases}$$

Since, C starts and ends at u, then

$$d(u) = 2 + 2 \times \begin{cases} \text{The no. of times } u \text{ occurs} \\ \text{inside the Euler cycle } C \end{cases}$$

$\therefore$ Every vertex of G has even degree.

Conversely, assume that every vertex of G has even degree.

To prove:

G is an Euler graph. suppose not, that is G has no Euler cycle.

Since every vertex of G has even degree (atleast 2) the G has a closed cycle.

Let C be a closed cycle of maximum

If C contains all the edges of G, then C itself a Eulerian cycle.

$\therefore E(G) - E(C)$ contains some other component G'.

Since all the vertices of G are of even degree and all the vertices of C are also even degree.

$\therefore$ Every vertex of G' also contains even degree (atleast 2).

$\therefore G'$ contains a cycle c'.

Since G is a connected graph, then c and c' should contain a common vertex w.

Now join c and c' using w.

Now $C \cup C'$ will be a new cycle in G and $E(C \cup C') > E(C)$ which is a $\Rightarrow\Leftarrow$ to C is a maximum cycle.

$\therefore G$ is an Euler graph.

Hence the proof.

---

7) If G is a simple graph with atleast 3 vertices and $\delta(G) \geq \left[\frac{V(G)}{2}\right]$ then G is Hamiltonian.

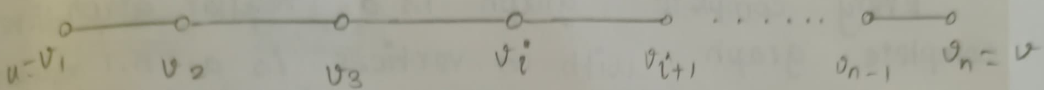
<u>Proof</u>: Let G be any connected graph with $n \geq 3$ vertices such that $\delta \geq n/2$.

Suppose G is not Hamiltonian and G is maximal non-Hamiltonian.

Since G is not complete. Let u and v be any two non-adjacent vertices in G.

By our choice of G, $G+uv$ is Hamiltonian. Also since G is non-Hamiltonian, each Hamilton cycle of $G+uv$ must contain the edge uv.

Thus there is a Hamilton path. $v_1, v_2, \ldots v_n$ in G with origin $u = v_1$ and end $v = v_n$



Let $S = \{v_i / uv_{i+1} \in E(G)\}$ and

$T = \{v_i / v_i v \in E(G)\}$

Since $v = v_n \notin S \cup T$ we have $|S \cup T| < n$

Also $|S \cap T| = 0$

Now $d(u) + d(v) = |S| + |T|$

$= |S \cup T| + |S \cap T|$

$\leq n + 0$

$< n$

$\therefore d(u) + d(v) < n$

But which is a $\Rightarrow\Leftarrow$ to $S \geq n/2$.

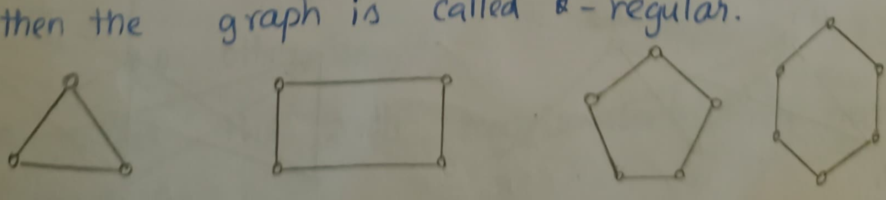
$\therefore G$ is Hamiltonian.

Hence the proof.

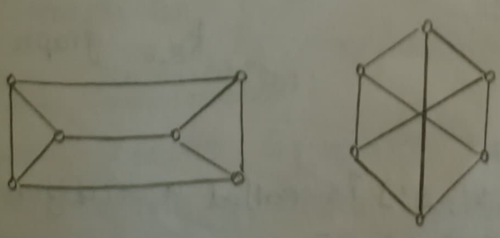
Regular graph:

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree k, then the graph is called k-regular.



→ 2-regular graph



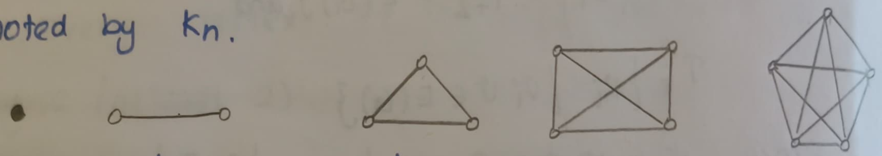
→ 3-regular graphs

Complete graph:

In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

Note:

Every complete graph is a regular graph. Every complete graph with n vertices is a $n-1$ regular graph. The complete graph with n vertices is denoted by K_n.

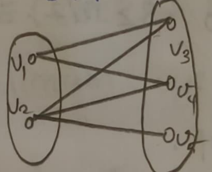


k_1 k_2 k_3 K_4 k_5

Bipartite graph:

A graph G is said to be bipartile if its vertex set $V(G)$ can be partitioned into two disjoint non-empty sets V_1 and V_2 $V_1 \cup V_2 = V(G)$ such that every edge in $E(G)$ has one end vertex in V_1 and another end vertex in V_2.

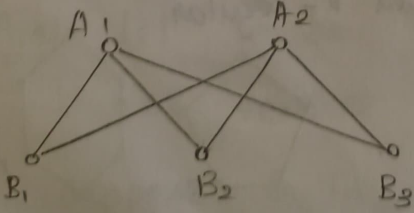
Eg:

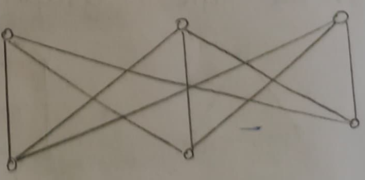


Complete bipartite graph:

A bipartite graph G with the bipartition V_1 and V_2 is called complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2.

A complete bipartite graph with m and n vertices in the bipartition is denoted by $K_{m,n}$.





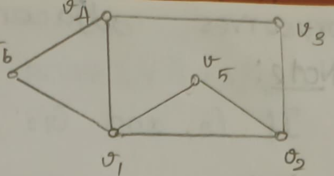
$k_{3,3}$ graph

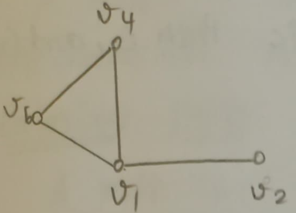
$k_{2,3}$ graph

Subgraph:

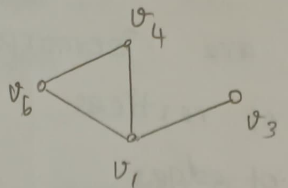
A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

E.g: consider the graph G





Subgraph of G



Not a subgraph of G

Adjacency matrix:

Let $G=(V,E)$ be a simple graph with n vertices $\{v_1, v_2, \ldots v_n\}$. Its adjacency matrix is denoted by $A=[a_{ij}]$ and defined by

$$A=[a_{ij}]=\begin{cases}1 & \text{if there exist an edge between } v_i \text{ and } v_j\\ 0 & \text{otherwise}\end{cases}$$

Note!

The adjacency matrix of a simple graph is symmetric.

Incidence matrix:

Let $G=(V,E)$ be an undirected graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $\{e_1, e_2, \ldots, e_m\}$. Then the $n\times m$ matrix $B=[b_{ij}]$ where

$$b_{ij}=\begin{cases}1 & \text{when edge } e_j \text{ incident on } v_i\\ 0 & \text{otherwise}\end{cases}$$

Path matrix!

If $G=(V,E)$ be a simple digraph in which $|V|=n$ and the nodes of G are assumed to be ordered. An $n\times n$ matrix P whose elements are given by,

$$P_{ij}=\begin{cases}1 & \text{if there exists a path from } v_i \text{ to } v_j\\ 0 & \text{otherwise}\end{cases}$$

is called the path matrix (reachability matrix) of the graph G.

Graph isomorphism:

Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

Note:

If G_1 and G_2 are isomorphic then G_1 and G_2 have

(i) The same no. of vertices

(ii) The same no. of edges

(iii) An equal no. of vertices with a given degree.

Results:

① Two graphs are isomorphic, iff their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

② Two simple graphs G_1 and G_2 are isomorphic iff their adjacency matrices A_1 and A_2 are related by $A_1 = P^{-1} A_2 P$. where P is a permutation matrix.

③ A matrix whose rows are the rows of the unit matrix, but not necessarily in their natural order, is called Permutation matrix.

Path:

A path in a graph is a sequence $v_1, v_2, \ldots, v_k$ of vertices each adjacent to the next.

Length of the path:

The no. of edges appearing in the sequence of a path is called the length of path.

Cycle or circuit:

A path which originates and ends in the same node is called a cycle or circuit.

Simple path:

A path is said to be simple if all the edges in the path are distinct.

Elementry path:

A path in which all the vertices are traversed only one is called an elementry path.

Connected path:

An directed graph is said to be connected if any pair of nodes are reachable from one another that is there is a path between any pair of nodes.

A graph which is not connected is called disconnected graph.

Eulerian path:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.

Eulerian cycle:

A cycle of a graph G is called an Eulerian cycle, if it includes each edge of G exactly once.

An Eulerian cycle should satisfies the following conditions.

(i) Starting and end points are same.

(ii) Cycle should contain all the edges of graph but exactly once.

Euler graph:

Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.

Hamiltonian path:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.

Hamiltonian cycle:

A cycle of a graph G is called a Hamiltonian cycle, if it includes each vertex of G exactly once, except the starting and ending vertices.

Hamiltonian graph:

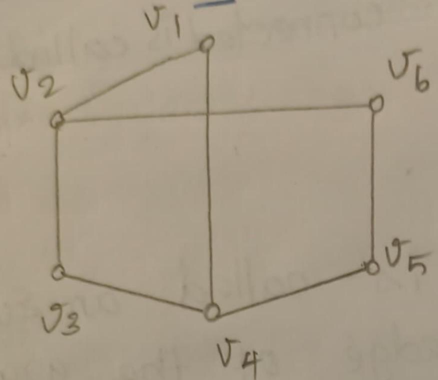
Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.

---

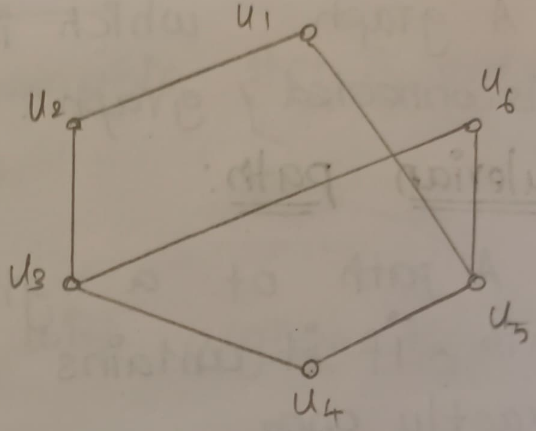
1) Examine whether the following graphs G and G' associated with the following adjacency matrices are isomorphic.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \text{ and } \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

G



G'



G	G'
(i) No. of vertices $n=6$	(i) No. of vertices $n=6$
(ii) No. of edges $e=7$	(ii) No. of edges $e=7$
(iii) Degree sequence 2, 3, 2, 3, 2, 2	(iii) Degree sequence 2, 2, 3, 2, 3, 2

Mapping:

$v_1 \rightarrow u_4$

$v_2 \rightarrow u_3$

$v_3 \rightarrow u_6$

$v_4 \rightarrow u_5$

$v_5 \rightarrow u_1$

$v_6 \rightarrow u_2$

Hence G and G' are isomorphic.

Unit - 1 Algebraic structures

Algebraic System:

A non-empty set G together with one or more n_ary operations say $*$ (binary) is called an Algebraic system or Algebraic structure or Algebra.

We denote it by $(G, *)$

Note:

$+, -, *, \cdot, \times, \cup, \cap$ etc are some of binary operations.

Properties of binary operations

Let the binary operations be $* : G \times G \to G$

(i) Closure Property:

$a*b = x \in G, \forall a, b \in G.$

(ii) Commutativity:

$a*b = b*a, \forall a, b \in G$

(iii) Associativity:

$(a*b)*c = a*(b*c), \forall a, b, c \in G$

(iv) Identity Element:

$a*e = e*a = a \quad \forall a \in G$, e is called the Identity element.

(v) Inverse element:

If $a*b = b*a = e$ (Identity) then b is called the inverse of a.

(vi) Distributive properties

$a*(b.c) = (a*b).(a*c)$ (Left distributive law)

$(b.c)*a = (b*a).(c*a)$ (Right distributive law) $\forall a, b \in G$

(vii) Cancellation Properties

$a*b = a*c \Rightarrow b = c$ (Left cancellation law)

$b*a = c*a \Rightarrow b = c$ (Right cancellation law) $\forall a, b, c \in G$

Notations:

Z - set of all integers

Q - The set of all rational numbers.

Q^+ - The set of all positive rational numbers

R - The set of all real numbers

R^+ - The set of all positive real numbers

C - The set of all complex numbers

N - The set of all natural numbers

Semigroup:

If a non-empty set S together with the binary operation $*$ satisfying the closure and associative commulative properties then $(S, *)$ is called a commutative semigroup.

Monoid:

A semigroup $(S, *)$ with an indentity element with respect to $*$ is called Monoid.

For example

Let $N = \{1, 2, 3, \ldots\}$ be the set of all natural numbers.

(i) **Closure property:**

Let $a, b \in N$

$a + b \in N$

(ii) **Associativity:**

Let $a, b, c \in N$

$(a+b)+c = a+(b+c)$

$\therefore (N, +)$ is a semigroup.

(iii) **Identity:**

Let $a \in N$

$a + 0 = a$

But $0 \notin N$

$(N, +)$ is not a monoid.

Group:

A non-empty set G together with the binary operation $*$ that is $(G, *)$ is a called group if $*$ satisfied following condition.

(i) Closure: $a * b \in G$, $\forall a, b \in G$.

(ii) Association: $(a*b)*c = a*(b*c)$, $\forall a, b, c \in G$.

(iii) Identity: There exists an element $e \in G$ called the identity element such that $a * e = e * a = a$, $\forall a \in G$.

(iv) Inverse: There exists an element $a^{-1} \in G$ called the inverse of a such that $a * a^{-1} = a^{-1} * a = e$ $\forall a \in G$.

Abelian group (or) Commutative group

In a group $(G, *)$ if $a*b = b*a$, $\forall a, b \in G$ then the group $(G, *)$ is called an abelian group otherwise $(G, *)$ is called non-abelian group.

Example:

$(z, +)$

$a + (-a) = 0$

$a + b = b + a$

Order of a group

The no. of elements in a group G is called, the order of the group and it is denote by $o(G)$ (or) $|G|$

finite and infinite group

If order of G is finite then G is set to be a finite group.

If order of G is infinite then G is set to be infinite group.

1) Show that $(Z, *)$ is a group where $*$ is defined by $a * b = a+b+1$.

Given $a*b = a+b+1$.

To prove: $(Z, *)$ is a group.

i) Closure:

Let $a, b \in Z$

$a*b = a+b+1 \in Z$.

(ii) Associative:

$(a*b)*c = a*(b*c)$

LHS $(a*b)*c = (a+b+1)*c$

$= a+b+1+c+1$

$= a+b+c+2$.

RHS $a*(b*c) = a*(b+c+1)$

$= a+b+c+1+1$

$= a+b+c+2$

$\therefore (a*b)*c = a*(b*c)$

(iii) Identity:

$a*e = a$

Let $a \in Z$

$a*e = a$

$a+e+1 = a$

$e = -1 \in Z$

(iv) Inverse:

$a*a^{-1} = e$

Let $a \in Z$

$a*a^{-1} = -1$

$a+a^{-1}+1 = -1$

$a+a^{-1} = -2$

$a^{-1} = -2-a$

Hence $(Z, *)$ is a group.

2) Prove that $[Z_5, +_5]$ is an abelian group.
Addition modulo 5

To prove: $[Z_5, +_5]$ is an abelian group.

$Z_5 = \{0, 1, 2, 3, 4\}$

Cayley Table:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

From the above table, we have closure, associative and commutative properties holds good.

Identity element is $0 \in Z_5$.

Inverse:

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1.

Hence $[Z_5, +_5]$ is an abelian group.

---

3) Prove that $G = \{[1], [2], [3], [4]\}$ is an abelian group under multiplication modulo 5:

To Prove: $G = \{[1], [2], [3], [4]\}$ is an abelian group.

$G = \{1, 2, 3, 4\}$

Cayley Table:

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

From the above table, we have closure, associative and commutative properties holds good.

Identity element is $1 \in G$.

Inverse:

Inverse of 1 is 1.

Inverse of 2 is 3.

Inverse of 3 is 2.

Inverse of 4 is 4.

$\therefore$ Hence, G is an abelian group.

4) Show that $(Q^+, *)$ is an abelian group, where * defined by $a*b = \frac{ab}{2}, \forall a,b \in Q^+$.

Given $a*b = \frac{ab}{2}, \forall a,b \in Q^+$

To Prove: $(Q^+,*)$ is an abelian group.

(i) Closure:

Let $a,b \in Q^+$.

$a*b = \frac{ab}{2} \in Q^+$.

(ii) Associative:

$(a*b)*c = a*(b*c)$

LHS $(a*b)*c = \frac{ab}{2} * c$

$= \frac{abc}{4}$

RHS

$a*(b*c) = a * \frac{bc}{2}$

$= \frac{abc}{4}$

$\therefore$ LHS = RHS.

(iii) Identity

$a*e = a$

Let $a \in Q^+$

$a*e = a$

$\frac{ae}{2} = a$

$e = \frac{2a}{a} = 2 \in Q^+$

$\therefore$ 2 is an identity element.

(iv) Inverse

$a*a^{-1} = e$.

Let $a \in Q^+$

$a*a^{-1} = 2$

$\frac{aa^{-1}}{2} = 2$

$a^{-1} = \frac{4}{a} \in Q^+$

(v) Commutative

$$a * b = b * a$$

$$a * b = \frac{ab}{2}$$

$$b * a = \frac{ba}{2} = \frac{ab}{2}$$

$$\therefore a * b = b * a$$

Hence $(Q^+, *)$ is an abelian group.

---

5) Examine whether $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} a \neq 0 \in R \right\}$ is a Abelian group commutative group under matrix multiplication, where R is the set of all real numbers.

Given $G = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : a \neq 0 \in \mathbb{R} \right\}$

To prove:

$(G, \cdot)$ is an abelian group

(i) Closure:

Let $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and $B = \begin{pmatrix} b & b \\ b & b \end{pmatrix}$ where $a \neq 0 \in R$ & $b \neq 0 \in R$

$$A \times B = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \in G.$$

(ii) Associative:

$$(A \times B) \times C = A \times (B \times C)$$

LHS

$$(A \times B) \times C = \left(\begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} b & b \\ b & b \end{pmatrix} \right) \times \begin{pmatrix} c & c \\ c & c \end{pmatrix}$$

$$= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \times \begin{pmatrix} c & c \\ c & c \end{pmatrix}$$

$$= \begin{pmatrix} 4abc & 4abc \\ 4abc & 4abc \end{pmatrix}$$

RHS

$$A \times (B \times C) = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \left(\begin{pmatrix} b & b \\ b & b \end{pmatrix} \times \begin{pmatrix} c & c \\ c & c \end{pmatrix} \right)$$

$$= \begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} 2bc & 2bc \\ 2bc & 2bc \end{pmatrix}$$

$$= \begin{pmatrix} 4abc & 4abc \\ 4abc & 4abc \end{pmatrix}$$

(iii) Identity

$$A \times E = A$$

Let $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \in G$

$$A \times E = A$$

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

$$\begin{pmatrix} 2ae & 2ae \\ 2ae & 2ae \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

$$2ae = a$$

$$e = \frac{a}{2a} = 1/2$$

$$\therefore E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in G.$$

(iv) Inverse:

$$A \times A^{-1} = E$$

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} a^{-1} & a^{-1} \\ a^{-1} & a^{-1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 2aa^{-1} & 2aa^{-1} \\ 2aa^{-1} & 2aa^{-1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$2aa^{-1} = 1/2$$

$$a^{-1} = \frac{1}{2 \times 2a} = \frac{1}{4a}$$

$$A^{-1} = \begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix} \in G.$$

(v) Commutative:

LHS: $= A \times B = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \times \begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix}$

$$= \begin{pmatrix} 2ba & 2ba \\ 2ba & 2ba \end{pmatrix} = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \times \begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

$= B \times A$

$= RHS$

$\therefore (G, \cdot)$ is an abelian group.

---

6) In any group $(G, *)$, show that $(a*b)^{-1} = b^{-1} * a^{-1}$, $\forall a, b \in G$.

Proof: Given $(G, *)$ is a group.

Let $a \in G$ and $b \in G$.

$\therefore a^{-1} \in G, b^{-1} \in G$.

$a * a^{-1} = a^{-1} * a = e$

$b * b^{-1} = b^{-1} * b = e$

$(a*b) * (b^{-1} * a^{-1}) = a * (b * (b^{-1} * a^{-1}))$

($\because$ Associative property)

$= a * ((b * b^{-1}) * a^{-1})$ ($\because$ Associative)

$= a * (e * a^{-1})$

$= a * a^{-1}$

$= e$

Similarly,

$(b^{-1} * a^{-1}) * (a*b) = e$

$\therefore (a*b)^{-1} = b^{-1} * a^{-1}$

$\therefore$ Hence the proof.

---

7) If every element in a group is its own inverse, then the group must be abelian

Proof: Let G be a group.

Let $a, b \in G \Rightarrow a * b \in G$.

Given every element in a group is its own inverse.

that is $a = a^{-1}$ and $b = b^{-1}$.

$$(a*b) = (a*b)$$

$= b^{-1} * a^{-1}$ ($\because$ by previous theorem)

$= b * a$ ($\because (a*b)^{-1} = b^{-1} * a^{-1}$)

Hence G is abelian.

$\therefore$ Hence the proof.

__Subgroup__

Let $(G,*)$ be a group. Then $(H,*)$ is said to be a subgroup of $(G,*)$ if $H \subseteq G$ and $(H,*)$ itself is a group under the operation $*$.

__Example:__

$(Q,+)$ is a subgroup of $(R,+)$

$(R,+)$ is a subgroup of $(C,+)$

__Note:__

For any group $(G,*)$

(i) the subgroups $(G,*)$ and $(\{e\},*)$ are called improper or trivial subgroups.

(ii) All the other groups are called the proper or nontrivial subgroups.

__Theorem:__

$\underline{\text{Sufficient \& necessary condition for subgroup}}$

A non-empty subset H of a group $(G,*)$ is a subgroup of G iff $a*b^{-1} \in H, \forall a, b \in H$.

__Proof:__

Assumes that H is a subgroup of G.

Since H itself a group, we have

$a, b \in H \Rightarrow a*b \in H$ (closure)

Since $b \in H \Rightarrow b^{-1} \in H$

$\therefore a, b \in H \Rightarrow a, b^{-1} \in H$

$\Rightarrow a*b^{-1} \in H$

Conversely, assume that $a*b^{-1} \in H, \forall a, b \in H$

__To prove:__ H is a subgroup of G.

(i) closure:

Let $b \in H \Rightarrow b^{-1} \in H$

$a, b \in H \Rightarrow a, b^{-1} \in H$

$a * (b^{-1})^{-1} \in H \Rightarrow a * b \in H$

$\therefore$ H is closed.

(ii) Associative:

Let $a, b, c \in H$

$(a * b) * c = a * (b * c)$

(iii) Identity:

Let $a \in H$

$\Rightarrow a^{-1} \in H$

$\Rightarrow a * a^{-1} \in H$

$\Rightarrow e \in H$

(iv) Inverse:

Let $a, e \in H$

$\Rightarrow e * a^{-1} \in H$

$\Rightarrow a^{-1} \in H$.

Every element of H has its inverse $a^{-1} \in H$.

Hence H is a subgroup of G.

Theorem:

The intersection of any 2 subgroups of a group $(G, *)$ is again a subgroup of $(G, *)$.

Proof:

Let H_1 and H_2 be subgroup of G.

$\therefore H_1 \cap H_2 \neq \phi$ ($\because$ atleast the identity element in H_1 and H_2)

Let $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$

$\Rightarrow a * b^{-1} \in H_1 \cap H_2$.

$\therefore a, b \in H_1 \cap H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgroup.

Homomorphism:

Let $(G,*)$ and (H,Δ) be any 2 groups. A mapping $f: G \to H$ is said to be homomorphism, if

$$f(a*b) = f(a) \Delta f(b) \text{ for any } a,b \in G.$$

Isomorphism:

A mapping 'f' from a group $(G,*)$ to a group (G',Δ) is said to be an isomorphism if,

i) f is a homomorphism that is
$$f(a*b) = f(a) \Delta f(b) \ \forall a,b \in G.$$

ii) f is 1-1 (Injective)

iii) f is onto (surjective)

In otherwords a bijective homomorphism is said to be an isomorphism.

Theorem:

Homomorphism preserves identities.

Proof:

Let $a \in G$

Let f be a homomorphism from $(G,*)$ into $(G',*)$

Clearly $f(a) \in G'$.

$f(a)*e' = f(a)$ ($\because e'$ is identity in G')

$= f(a*e)$ ($\because e$ is identity in G)

$= f(a)*f(e)$ ($\because f$ is homomorphism)

$\therefore f(a)*e' = f(a)*f(e)$

$\Rightarrow e' = f(e)$ ($\because$ Left cancellation law)

$\therefore f$ preserves identities.

Theorem:

Homomorphism preserves inverses.

Proof:

Let $a \in G$.

Since G is a group, $a^{-1} \in G$

$\because G$ is a group $a * a^{-1} = a^{-1} * a = e$.

$\therefore e' = f(e)$

$= f(a * a^{-1})$

$= f(a) * f(a^{-1})$ ($\because f$ is homomorphism)

$\therefore f(a) * f(a^{-1}) = e'$

$\therefore f(a^{-1})$ is the inverse of $f(a) \in G'$

$\therefore [f(a)]^{-1} = f(a^{-1})$

Cayley's theorem:

Every finite group of order n is isomorphic to permutation group of degree n.

Proof:

Step 1: We shall first find a set of G' of permutation

Step 2: We prove G' is a group.

Step 3: We prove $\phi: G \to G'$ is an isomorphism.

Step 1: Let G be a finite group of order n.

Let $a \in G$.

Define $f_a : G \to G$ by $f_a(x) = ax$.

Since $f_a(x) = f_a(y)$

$\Rightarrow ax = ay$

$\Rightarrow x = y$

$\therefore f_a$ is 1-1.

Since if $y \in G$, then

$f_a(a^{-1}y) = aa^{-1}y = y$

$\therefore f_a$ is onto.

$\therefore f_a$ is bijection.

Since G has n elements f_a is just a permutation on n symbols.

Let $G' = \{f_a / a \in G\}$

Step 2:

To prove: G' is a group.

Let $f_a, f_b \in G'$

$f_a \circ f_b(x) = f_a(f_b(x)) = f_a(bx)$

$= abx$

$= f_{ab}(x)$

Hence $f_a \circ f_b = f_{ab}$

Hence G' is closed.

$f_e \in G'$ is the identity element.

The inverse of f_a in G' is f_a^{-1}

$\therefore G'$ is a group.

Step 3:

To prove:

G and G' are isomorphic.

Define $\phi : G \to G'$ by $\phi(a) = f_a$

$\phi(a) = \phi(b) \Rightarrow f_a = f_b$

$\Rightarrow f_a(x) = f_b(x)$

$\Rightarrow ax = bx$

$\Rightarrow a = b$

Hence ϕ is 1-1.

Since f_a is onto, ϕ is onto

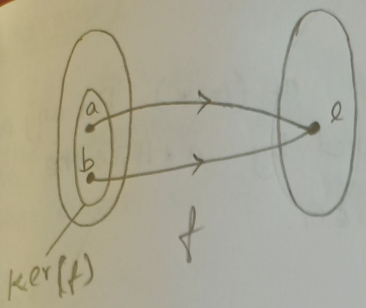
Also $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \cdot \phi(b)$

$\therefore \phi : G \to G'$ is an is isomorphism.

$\therefore G \cong G'$.

Kernel:

Let $f: G \to G'$ be a group homomorphism. The set of all elements of G which are mapped into e' (identity in G') is called the kernel of f and it is denoted by $\ker(f)$. that is $\ker(f) = \{x \in G \,/\, f(x) = e'\}$



Then $ker(f) = \{a, b\}$

E.g:

① $f : (Z, +) \to (Z, +)$ defined by $f(x) = 2x$.

Then $ker(f) = \{0\}$

② $f : (R, \cdot) \to (R^+, \cdot)$ defined by $f(x) = |x|$

Then $ker(f) = \{1, -1\}$

Theorem:

The kernel of a homomorphism f from a group $(G, *)$ to $(G', *)$ is a subgroup of G.

Proof:

We know that $ker(f) = \{x \in G / f(x) = e'\}$

Since $f(e) = e'$ is always true, atleast $e \in ker(f)$

Let $a, b \in ker(f)$

$\therefore f(a) = e'$ and $f(b) = e'$

$$f(a * b^{-1}) = f(a) * f(b^{-1})$$
$$= f(a) * [f(b)]^{-1}$$
$$= e' * e'$$
$$= e'$$
$$\Rightarrow a * b^{-1} \in ker(f)$$

$\therefore a, b \in ker(f) = a * b^{-1} \in ker(f)$

$\therefore ker(f)$ is a subgroup of G.

Hence the proof.

Left coset of H in G:

Let $(H, *)$ be a subgroup of $(G, *)$. For any $a \in G$, the left coset of H, denoted by $a * H$, is the set.

$$a * H = \{a * h : h \in H\} \ \forall a \in G.$$

Right coset of H in G:

The right coset of H, denoted by $H * a$, is the set

$$H * a = \{h * a : h \in H\} \ \forall a \in G.$$

Theorem:

Any two right (or left) cosets of H in G are either disjoint or identical.

Proof:

Let $H * a$ and $H * b$ be a right cosets of a subgroup H of G.

Let $a, b \in G$

We have to prove that either

$$(H * a) \cap (H * b) = \phi \quad \text{(or)} \quad H * a = H * b$$

Suppose, $(H * a) \cap (H * b) \neq \phi$ then there exists an element $x \in (H * a) \cap (H * b)$

$\Rightarrow x \in H * a$ and $x \in H * b$

$x \in H * a \Rightarrow H * x = H * a$ — (1)

and $x \in H * b \Rightarrow H * x = H * b$ — (2)

From (1) and (2),

$$H * x = H * a = H * b$$

$$\therefore H * a = H * b$$

Hence either $(H * a) \cap (H * b) = \phi$ (or) $H * a = H * b$

Hence the proof.

Result:

If $a \in H * b$ then $H * a = H * b$ and if $a \in b * H$, then $a * H = b * H$.

Lagrange's Theorem:

Let G be a finite group of order n and H be any subgroup of G. Then the order of H decides the order of G. that is $O(H)/O(G)$.

Proof:

Let $(G,*)$ be a group where order is n that is $O(G)=n$.

Let $(H,*)$ be a subgroup of G whose order is m that is $O(H)=m$.

Let $h_1, h_2, \ldots h_m$ be the m different elements of H.

The right coset $H*a$ of H in G is defined by

$H*a = \{h_1*a, h_2*a, \ldots, h_m*a\}, a\in G$.

Since there is a 1-1 correspondence between the elements of H and $H*a$, the elements of $H*a$ are distinct.

Hence each right coset of H in G has m distinct elements.

We know that, any right cosets of H in G are either disjoint or identical.

The number of distinct right cosets of H in G is finite (say k) ($\because G$ is finite)

The union of those k distinct cosets of H in G is equal to G.

Let there k distinct right cosets be $H*a_1$, $H*a_2, \ldots H*a_k$.

Then $G = (H*a_1) \cup (H*a_2) \cup \ldots \cup (H*a_k)$

$\therefore O(G) = O(H*a_1) + O(H*a_2) + \ldots + O(H*a_k)$

$n = m+m+\ldots+m$ (k times)

$n = km$

$\frac{n}{m} = k \Rightarrow \frac{O(G)}{O(H)} = k.$

Since k is an integer, $O(H)/O(G)$

Hence the proof.

Normal subgroups:

Let H be a subgroup of G under $*$. Then H is said to be a normal subgroup of G, for every $x \in G$ and for $h \in H$, if $x * h * x^{-1} \in H$, $x * H * x^{-1} \subseteq H$.

(or) A subgroup H of G is called a normal subgroup of G if $x * h = h * x$, $\forall x \in G$.

Theorem:

The intersection of any 2 normal subgroups of a group is a normal subgroup.

Proof:

Given H and k are normal subgroups.

$\Rightarrow H$ and k are subgroups of G.

$\Rightarrow H \cap k$ ~~are~~ is a subgroup of G.

To prove:

$H \cap k$ is normal.

Let $x \in G$ and $h \in H \cap k$

$x \in G$ and $h \in H$ and $h \in k$

$x \in G$, $h \in H$ and $x \in G$, $h \in k$.

$\underset{(1)}{x * h * x^{-1} \in H}$ and $x * h * x^{-1} \in k$ — (2) ($\because$ H and k are normal subgroups)

From (1) and (2),

$x * h * x^{-1} \in H \cap k$

$\Rightarrow H \cap k$ is a normal subgroup of G.

Hence the proof.

Fundamental theorem on Homomorphism of groups:

Every homomorphic image of a group G is isomorphic to some quotient group of G. (or) Let $f: G \to G'$ be onto homomorphism of groups with kernel k. Then $G/k \cong G'$.

Proof:

Let f be the homomorphism $f: G \to G'$ and let k be the kernel of this homomorphism.

claim: $G/k \cong G'$

Define $\phi: G/k \to G'$ by $\phi(k*a) = f(a), \forall a \in G$.

To prove: ϕ is well defined.

We have $k*a = k*b$

$\Rightarrow a*b^{-1} \in k$

$\Rightarrow f(a*b^{-1}) = e' \Rightarrow f(a)*f(b^{-1}) = e'$.

$\Rightarrow f(a) * [f(b)]^{-1} = e'$

$\Rightarrow f(a) * [f(b)]^{-1} * f(b) = e' * f(b)$ $\quad (\because f(b) * f(b^{-1}) = e')$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \phi(k*a) = \phi(k*b)$

$\therefore \phi$ is well-defined.

To prove: ϕ is 1-1.

$\phi(k*a) = \phi(k*b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f(a) * f(b^{-1}) = f(b) * f(b^{-1})$

$\Rightarrow f(a) * f(b^{-1}) = f(b*b^{-1}) = f(e) = e'$.

$\Rightarrow f(a*b^{-1}) = e'$

$\Rightarrow a*b^{-1} \in k$

$\Rightarrow k*a = k*b$

$\therefore \phi$ is 1-1.

To prove: ϕ is onto.

Let $y \in G'$.

Since f is onto, $\exists\, a \in G$ such that $f(a) = y$.

Hence $\phi(k*a) = f(a) = y$

$\therefore \phi$ is onto.

To prove: ϕ is a homomorphism.

$\phi(k*a*k*b) = \phi(k*a*b)$

$= f(a*b) = f(a)*f(b)$

$= \phi(k*a) * \phi(k*b)$

$\therefore \phi$ is homomorphism.

Since ϕ is 1-1, onto and homomorphism. ϕ is an isomorphism between G/K and G'. $\therefore G/K \cong G'$.

Hence the proof.

Definition:

Let G be a group. Let $a \in G$. Then $H = \{a^n / n \in \mathbb{Z}\}$ is a subgroup of G. H is called the cyclic subgroup of G generated by a and is denoted by $\langle a \rangle$.

Let G be a group and let $a \in G$. 'a' is called a generator of G if $\langle a \rangle = G$. A group G is cyclic if there exists an element $a \in G$ such that $\langle a \rangle = G$

Note:

If G is a cyclic group generated by an element a, then every element of G is of the form a^n for some $n \in \mathbb{Z}$.

Theorem:

Every cyclic group is an abelian group.

Proof:

Let $G = \langle a \rangle$ be a cyclic group.

Let $x, y \in G$.

Then $x = a^r$ and $y = a^s$ for some $r, s \in \mathbb{Z}$.

Hence $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = yx$.

$\therefore G$ is abelian.

Hence the proof.

Theorem:

Every subgroup of a cyclic group is cyclic.

Proof:

Let G be a cyclic group generated by a and let H be a subgroup of G.

To prove: H is cyclic.

Clearly every element of H is of the form a^n for some integer n.

Let m be the smallest positive integer such that $a^m \in H$. We claim that a^m is a generator of H.

Let $b \in H$. Then $b = a^n$ for some $n \in Z$.

Let $n = mq + r$ where $0 \leq r < m$.

Then $b = a^n = a^{mq+r}$

$= a^{mq} a^r = (a^m)^q a^r$

$\therefore a^r = (a^m)^{-q} b \quad -(1)$

Now, $a^m \in H$.

Since H is a subgroup, $(a^m)^{-q} \in H$.

Also $b \in H$.

By (1), $a^r \in H$, and $0 \leq r < m$.

But m is the least positive integer such that $a^n \in H$.

$\therefore r = 0$.

Hence $b = a^n = a^{qm} = (a^m)^q$

$\therefore$ Every element of H is a power of a^m.

$\therefore H = \langle a^m \rangle$ and hence H is cyclic.

Hence the proof.

Ring:

An algebraic system $(R, +, \cdot)$ is called a ring if the binary operations $+$ and $\cdot$ satisfies the following conditions:

(i) $(a+b)+c = a+(b+c)$, $a, b, c \in R$.

(ii) $\exists$ an element $0 \in R$ called zero element such that $a + 0 = 0 + a = a$, $\forall a \in R$.

(iii) For all $a \in R$, $a + (-a) = (-a) + a = 0$, $-a$ is the negative of a.

(iv) $a + b = b + a$, $\forall a, b \in R$.

(v) $(a.b).c = a.(b.c)$, $\forall a, b, c \in R$.

(vi) The operation $\cdot$ is distributive over $+$. that is for any $a, b, c \in R$.

$a.(b+c) = a.b + a.c$

$(b+c).a = b.a + c.a$

Eg:

$(Z, +, \cdot)$, $(R, +, \cdot)$, $(Q, +, \cdot)$, $(C, +, \cdot)$ are rings.

Field:

A commutative ring with identity $(R,+,\cdot)$ is called a field if every non-zero element has a multiplicative inverse.

Thus $(R,+,\cdot)$ is a field if

(i) $(R,+)$ is an abelian group and.

(ii) $(R-\{0\},\cdot)$ is also an abelian group.

E.g:

$(\mathbb{R},+,\cdot)$ and $(\mathbb{Q},+,\cdot)$ are fields. $(\mathbb{Z},+,\cdot)$ is not a field

Unit-5

LATTICES AND BOOLEAN ALGEBRA

Relation:

A relation R on a set A is a well defined rule, which tells whether given 2 elements x and y of A are related or not.

If x is related to y, we write xRy. otherwise $x \not R y$.

Note:

If A is a finite set with n elements then $A \times A$ has n^2 elements.

$\therefore A \times A$ has 2^{n^2} relation on an n element set.

R-is reflexive:

Let X be a set. R be a relation defined on X. Then R is said to be reflexive if it satisfies the following condition.

$$xRx, \forall x \in X.$$

R-is reflexive $= \{x / (x, x) \in R\} \forall x \in R$

R-is symmetric:

Let X be any set. R be a relation defined on X. Then R is said to be symmetric, if it satisfies the following condition.

$$xRy \Rightarrow yRx, \forall x, y \in X.$$

R-is symmetric $= \{(y, x) / (x, y) \in R \Rightarrow (y, x) \in R\}, \forall x, y \in X.$

Note:

A relation which is not symmetric is called Asymmetric

R-is transitive:

Let x be any set. R be a relation defined on x. Then R is said to be transitive, if R satisfies the following condition.

$$xRy \text{ and } yRz \Rightarrow xRz, \forall x, y, z \in X.$$

R is transitive $= \{(x, z) / (x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R\}$

$\forall x, y, z \in X.$

R is Antisymmetric:

Let x be any set. R be a relation defined on x. Then R is said to be antisymmetric if it satisfies the following condition.

$$xRy \text{ and } yRx \Rightarrow x = y, \ \forall x, y \in X.$$

Equivalence relation:

Let x be any set. R be a relation defined on x. If R satisfies reflexive, symmetric and transitive then the Relation R is said to be an equivalence relation.

Partial order relation:

Let x be any set. R be a relation defined on x. Then R is said to be a partial order relation if it satisfies reflexive, antisymmetric and transitive relation.

Partially ordered set or Poset:

A set together with a partial order relation defined on it is called partially ordered set or Poset.

Upper bound and Lower bound:

Let $(P, \leq)$ be a Poset and A be any non-empty subset of P. An element $a \in P$ is an upper bound of A, if $a \geq x, \forall x \in A$.

An element $b \in P$ is said to be lower bound for A, if

$$b \leq x, \forall x \in A.$$

Least upper bound (LUB):

Let $(P, \leq)$ be a Poset and $A \subseteq P$. An element $a \in P$ is said to be least upper bound (LUB) or supremum (sup) of A, if

(i) a is a upper bound of A.

(ii) $a \leq c$, where c is any other upper bound of A.

Greatest Lower bound (GLB):

Let $(P, \leq)$ be a Poset and $A \subseteq P$. An element $b \in P$ is said to be greatest lower bound (GLB) or infimum (inf) of A, if

(i) b is a lower bound of A.

(ii) $b \geq d$, where d is any other lower bound of A.

Lattice:

A lattice is a partially ordered set (Poset) $(L, \leq)$, In which for every pair of elements $a, b \in L$, both greatest lower bound (GLB) and least upper bound (LUB) exists.

Note:

$GLB\{a, b\} = a * b$ (or) $a \wedge b$ (or) $a \cdot b$

$LUB\{a, b\} = a \oplus b$ (or) $a \vee b$ (or) $a + b$

---

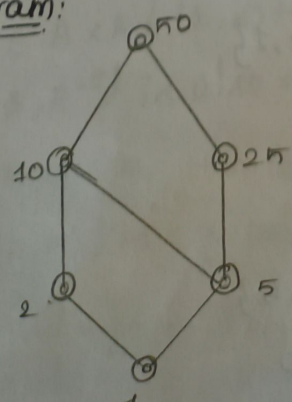
1) Consider the set $D_{50} = \{1, 2, 5, 10, 25, 50\}$ and the relation divides (/) be a partial ordering relation on D_{50}.
(i) Draw the Hasse diagram of D_{50} with relation divides (ii) Determine all upper bounds of 5 and 10 (iii) Determine all lower bounds of 5 and 10 (iv) Determine LUB of 5 and 10 (v) Determine GLB of 5 and 10.

Given $D_{50} = \{1, 2, 5, 10, 25, 50\}$

$R = \{\langle a, b\rangle / a | b\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 5,5\rangle, \langle 10,10\rangle, \langle 25,25\rangle, \langle 50,50\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 1,10\rangle, \langle 1,25\rangle, \langle 1,50\rangle, \langle 2,10\rangle, \langle 2,50\rangle, \langle 5,10\rangle, \langle 5,25\rangle, \langle 5,50\rangle, \langle 10,50\rangle, \langle 25,50\rangle\}$

(i) Hasse diagram:



(ii) UB $\{5,10\} = \{10,50\}$

(iii) LB $\{5,10\} = \{5,1\}$

(iv) LUB $\{5,10\} = 10$

(v) GLB $\{5,10\} = 5$

---

2) Let $D_{100} = \{1,2,4,5,10,20,25,50,100\}$ be the divisors of 100. Draw the Hasse diagram of $(D_{100}, /)$ where / is the relation "division". Find (i) glb $\{10,20\}$ (ii) lub $\{10,20\}$ (iii) glb $\{5,10,20,25\}$ (iv) lub $\{5,10,20,25\}$

Given $D_{100} = \{1,2,4,5,10,20,25,50,100\}$

$R = \{\langle a,b\rangle / a|b\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 4,4\rangle, \langle 5,5\rangle, \langle 10,10\rangle, \langle 20,20\rangle, \langle 25,2$

$\langle 50,50\rangle, \langle 100,100\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 1,10\rangle,$

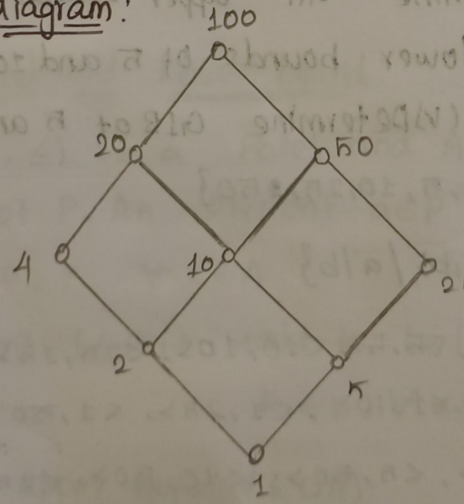
$\langle 1,20\rangle, \langle 1,25\rangle, \langle 1,50\rangle, \langle 1,100\rangle, \langle 2,4\rangle, \langle 2,10\rangle, \langle 2,2$

$\langle 2,50\rangle, \langle 2,100\rangle, \langle 4,20\rangle, \langle 4,100\rangle, \langle 5,10\rangle, \langle 5,20\rangle, \langle 5,25\rangle$

$\langle 5,50\rangle, \langle 5,100\rangle, \langle 10,20\rangle, \langle 10,50\rangle, \langle 10,100\rangle, \langle 20,100\rangle,$

$\langle 25,50\rangle, \langle 25,100\rangle, \langle 50,100\rangle\}$

Hasse diagram:



(i) glb $\{10,20\} = 10$

lb $\{10,20\} = \{10,5,2,1\}$

(ii) lub $\{10,20\} = 20$

ub $\{10,20\} = \{20,100\}$

(iii) glb $\{5,10,20,25\} = 5$

lb $\{5,10,20,25\} = \{1,5\}$

(iv) lub $\{5,10,20,25\} = 100$

ub $\{5,10,20,25\} = 100$

3) Show that $(N, \leq)$ is a partially ordered set where N is the set of all positive integers and $\leq$ is defined by $m \leq n$ iff $n-m$ is a non-negative integer.

Given N is the set of all positive integer.

The given relation is $m \leq n$ iff $n-m$ is a non-negative integer.

Now, $\forall x \in N$

$x = x \Rightarrow x - x = 0$ is a non-negative integer.

$\therefore xRx, \forall x \in N.$

$\therefore R$ is reflexive.

consider, xRy and yRx.

$x \leq y \Rightarrow y-x$ is a non-negative integer —(1).

$y \leq x \Rightarrow x-y$ is a non-negative integer.

$\Rightarrow -(y-x)$ is a non-negative integer—(2)

From (1) and (2), we get $x = y$.

$\therefore R$ is Antisymmetric.

Assume xRy and yRz.

$x \leq y \Rightarrow y-x$ is a non-negative integer.

$y \leq z \Rightarrow z-y$ is a non-negative integer.

$\Rightarrow (y-x)+(z-y)$ is a non-negative integer.

$\Rightarrow z-x$ is a non-negative integer.

$\Rightarrow x \leq z$.

$\therefore xRy$ and $yRz \Rightarrow xRz$.

$\therefore R$ is transitive. Hence R is a partial order relation.

$\therefore (N, \leq)$ is a partially ordered set.

---

4) Let R be a relation on a set A. Then define $R^{-1} = \{(a,b) \in A \times A \,/\, (b,a) \in R\}$. Prove that if (A, R) is Poset then (A, R^{-1}) is also a Poset.

Given A is a finite set.

$R = \{(b,a)\}$ is a partial order relation on A.

<u>Claim</u>:

$R^{-1} = \{(a,b)\}$ is a partial order relation.

Since $(a,a) \in R$.

$\Rightarrow R^{-1} = \{(a,a)\}$ is a reflexive.

Given R is Antisymmetric.

$(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$ — (1)

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

$(b,a) \in R \Rightarrow (a,b) \in R^{-1}$

$(a,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow a = b$ ($\because$ by using (1))

$\therefore R^{-1}$ is Antisymmetric.

Given R is transitive.

$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ — (2)

Since $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

and $(b,c) \in R \Rightarrow (c,b) \in R^{-1}$

Since $(c,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$ ($\because$ by using (2))

$\therefore R^{-1}$ is transitive.

Since R^{-1} is reflexive, Antisymmetric and transitive, R^{-1} is partial order relation.

$\therefore (A, R^{-1})$ is a Poset.

---

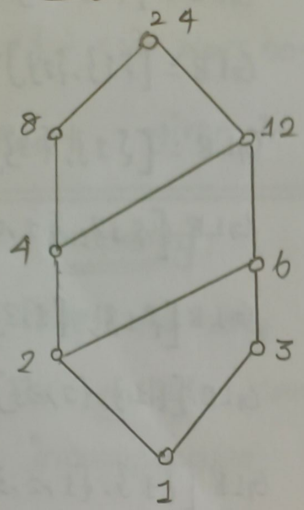
5) If Sn is the set of all divisors of the positive integers n and aDb iff a divides b, Prove that $\{S_{24}, D\}$ is a lattice.

$S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$R = \{\langle a,b \rangle / a|b\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 6,6\rangle, \langle 8,8\rangle, \langle 12,12\rangle, \langle 24,24\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,6\rangle, \langle 1,8\rangle, \langle 1,12\rangle, \langle 1,24\rangle, \langle 2,4\rangle, \langle 2,6\rangle, \langle 2,8\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 4,8\rangle, \langle 4,12\rangle, \langle 4,24\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 8,24\rangle, \langle 12,24\rangle\}$

Hasse diagram:



$UB = \{2,3\} = \{6, 12, 24\}$

$LUB = \{2,6\} = 6$

$UB = \{4, 12\} = \{12, 24\}$

$LUB\ \{4, 12\} = 12$

$LB\ \{2, 3\} = \{1\}$

$GLB\ \{2, 3\} = 1$

$LB\ \{4, 12\} = \{4, 2, 1\}$

$GLB\ \{4, 12\} = 4$

Similarly, every pair of elements in S_{24} have the GLB and LUB. Hence (S_{24}, D) is a lattice.

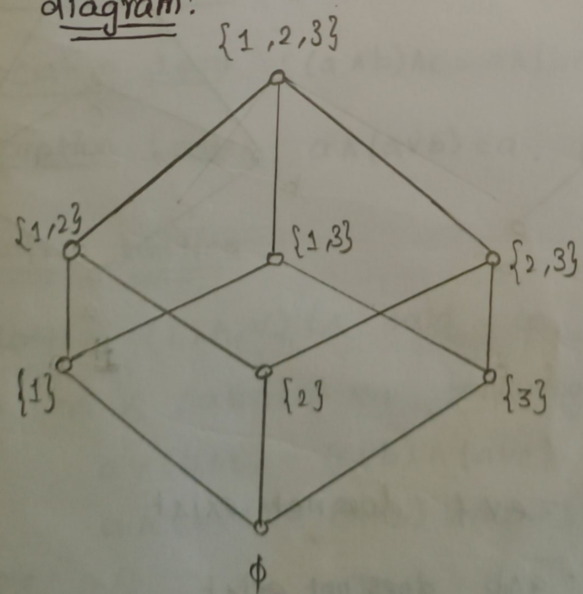
---

6) Determine whether $(P(A), \subseteq)$ where $A = \{1,2,3\}$ is a Lattice.

$A = \{1,2,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Hasse diagram:



$LUB[\{1\}, \phi] = \{1\}$ $\quad$ $GLB = [\{1\}, \phi] = \phi$

$LUB[\{1\}, \{2\}] = \{1,2\}$ $\quad$ $GLB = [\{1\}, \{2\}] = \phi$

$LUB[\{1\}, \{3\}] = \{1,3\}$ $\quad$ $GLB = [\{1\}, \{3\}] = \phi$

$LUB[\{1\}, \{1,2\}] = \{1,2\}$ $\quad$ $GLB[\{1\}, \{1,2\}] = \{1\}$

$LUB[\{1\}, \{1,3\}] = \{1,3\}$ $\quad$ $GLB[\{1\}, \{1,3\}] = \{1\}$

$LUB[\{1\}, \{2,3\}] = \{1,2,3\}$ $\quad$ $GLB[\{1\}, \{2,3\}] = \phi$

$LUB[\{1\}, \{1,2,3\}] = \{1,2,3\}$ $\quad$ $GLB[\{1\}, \{1,2,3\}] = \{1\}$

Similarly, we can easily verify both GLB and LUB exist for each pair of $P(A)$.

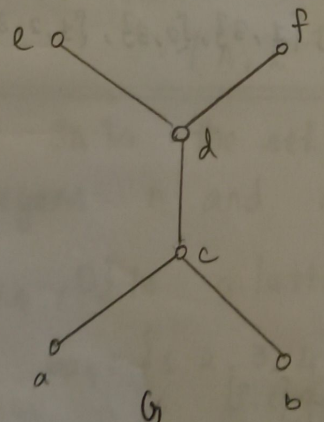
$\therefore (P(A), \subseteq)$ is a Lattice.

Note:

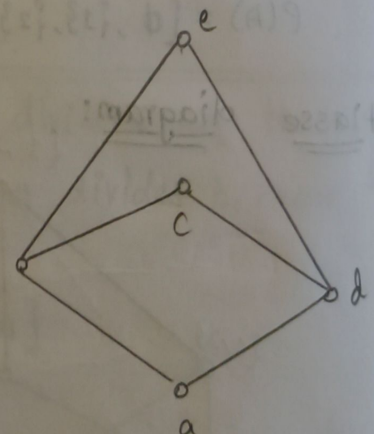
For any two subsets A and B of $P(A)$

$$LUB\{A,B\} = A \cup B$$
$$GLB\{A,B\} = A \cap B$$

7) Determine whether the Posets represented by each of the Hasse diagram given in figure are Lattice



G



H

In Fig G

$LUB\{e,f\} = e \vee f$ does not exist

$GLB\{a,b\} = a \wedge b$ does not exist

$\therefore$ The Poset given in Fig G is not a Lattice.

In Fig H

LUB$\{c,e\}$ = $c \vee e$ does not exist

$\therefore$ The Poset given in Fig H is not a Lattice

Comparable property:

In a Poset for any 2 elements a, b either $a \leq b$ or $b \leq a$ is called comparable property. Otherwise it is called incomparable property.

Claim:

A partially ordered set (Poset) $(P, \leq)$ is said to be totally ordered set or linearly oredered set or chain if any elements are comparable, that is given any 2 elements x and y of a poset either $x \leq y$ or $y \leq x$.

Properties of Lattice:

Let $(L, \wedge, \vee)$ be a given Lattice. Then $\wedge$ and $\vee$ satisfies the following conditions, $\forall a, b, c \in L$.

1. Idempodent Law : $a \wedge a = a$, $a \vee a = a$
2. Commutative Law : $a \wedge b = b \wedge a$, $a \vee b = b \vee a$
3. Associative Law : $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$
4. Absorption Law : $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$

Distributive Lattice

A Lattice $(L, \wedge, \vee)$ is said to be distributive Latice if $\wedge$ and $\vee$ satisfies the following conditions, $\forall a, b, c \in L$.

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Example:

$(P(A), \vee, \wedge)$ is a distributive Lattice.

Theorem:

Any chain is a distributive Lattice.

Proof:

Let $(L, \wedge, \vee)$ be a given chain and $\forall a, b \in L$.

Since any 2 elements of a chain are comparable, we have either $a \leq b$ or $b \leq a$.

Case (i): $a \leq b$

Then $GLB\{a, b\} = a$

$LUB\{a, b\} = b$

Case (i): $b \leq a$

Then $GLB\{a, b\} = b$

$LUB\{a, b\} = a$

In both cases, any 2 elements of a Chain has both GLB and LUB.

$\therefore$ Any chain is a Lattice.

Next, we prove $(L, \wedge, \vee)$ satisfies distributive property.

Let $a, b, c \in L$.

Since any chain satisfies comparable property, we have the following 6 cases.

Case (i) $a \leq b \leq c$ Case (ii) $a \leq c \leq b$ case (iii): $b \leq a \leq c$

case (iv) $b \leq c \leq a$ case (v) $c \leq a \leq b$ case (vi) $c \leq b \leq a$

Case (i): $a \leq b \leq c$

To prove: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

LHS

$a \vee (b \wedge c)$

$= a \vee b$

$= b$

RHS

$(a \vee b) \wedge (a \vee c)$

$= b \wedge c$

$= b$

$\therefore LHS = RHS$

To prove: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

LHS:

$a \wedge (b \vee c) = a \wedge c = a$

RHS: $(a \wedge b) \vee (a \wedge c) = a \vee a = a$

$\therefore LHS = RHS$

Case (ii): $a \le c \le b$

To prove: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

LHS $a \vee (b \wedge c) = a \vee c = c$

RHS $(a \vee b) \wedge (a \vee c) = b \wedge c = c$

$\therefore$ LHS = RHS.

To prove: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

LHS: $a \wedge (b \vee c) = a \wedge b = a$

RHS: $(a \wedge b) \vee (a \wedge c) = a \vee a = a$

$\therefore$ LHS = RHS.

Similarly, we can easily prove the distributive properties for the remaining 4 cases.

$\therefore (L, \wedge, \vee)$ is a distributive Lattice.

Hence any chain is a distributive Lattice. Hence the proof.

Theorem:

Let $(L, \wedge, \vee)$ be Lattice in which $\wedge$ and $\vee$ denote the operation of meet and join respectively. For any $a, b \in L$, $a \le b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$.

Proof: (i) $\Rightarrow$ (ii)

Let $a \le b$

From the defination of $a \vee b$, we have

$a \vee b \le b$ — (1)

Since $a \vee b$ is the LUB $\{a, b\}$, we have

$b \le a \vee b$ — (2)

From (1) and (2), we have $a \vee b = b$

(ii) $\Rightarrow$ (iii)

Let $a \vee b = b$

Now $a \wedge b = a \wedge (a \vee b) = a$ ($\because$ by Absorption Law)

$\therefore a \wedge b = a$

(iii) => (i)

Let $a \wedge b = a$. Then $GLB\{a,b\} = a \Rightarrow a \leq b$.

Hence the proof.

Theorem:

State and prove distributive inequalities of lattice

Statement:

Let $(L, \wedge, \vee)$ be a given Lattice. For any $a,b,c \in L$, the following inequality holds.

(i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Proof:

(i) claim: $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Since $b \wedge c = glb\{b,c\}$ $\quad$ $a \vee b = lub\{a,b\}$

$b \wedge c \leq b$ — (1) $\quad$ $a \leq a \vee b$ — (3)

$b \wedge c \leq c$ — (2) $\quad$ $b \leq a \vee b$ — (4)

From (1) and (4), $b \wedge c \leq b \leq a \vee b \Rightarrow b \wedge c \leq a \vee b$ — (5)

From (3) and (5), $a \vee (b \wedge c) \leq a \vee b$ — (A)

Since $a \vee c = lub\{a,c\}$

$a \leq a \vee c$ — (6)

$c \leq a \vee c$ — (7)

From (2) and (7), $b \wedge c \leq c \leq a \vee c \Rightarrow b \wedge c \leq a \vee c$ — (8)

From (6) and (8), $a \vee (b \wedge c) \leq a \vee c$ — (B)

From (A) and (B), $(a \vee b) \wedge (a \vee c) = glb\{(a \vee b), (a \vee c)\}$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

(ii) claim: $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Since $b \vee c = lub\{b,c\}$ $\quad$ $a \wedge b = glb\{a,b\}$ $\quad$ $a \wedge c = glb\{a,c\}$

$b \leq b \vee c$ — (1) $\quad$ $a \wedge b \leq a$ — (3) $\quad$ $a \wedge c \leq a$ — (6)

$c \leq b \vee c$ — (2) $\quad$ $a \wedge b \leq b$ — (4) $\quad$ $a \wedge c \leq c$ — (7)

From (1) and (4), $a \wedge b \leq b \leq b \vee c \Rightarrow a \wedge b \leq b \vee c$ — (5)

From (3) and (5), $a \wedge b \leq a \wedge (b \vee c)$ — (A)

From (2) and (7), $a \wedge c \leq c \leq b \vee c \Rightarrow a \wedge c \leq b \vee c$ — (8)

From (6) and (8), $a \wedge c \leq a \wedge (b \vee c)$ — (B)

From (A) and (B), $a \wedge b \leq a \wedge (b \vee c)$, $a \wedge c \leq a \wedge (b \vee c)$

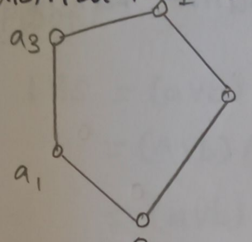
$(a \wedge b) \vee (a \wedge c) = \text{lub}\{(a \wedge b), (a \wedge c)\} \Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

$\therefore\ a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

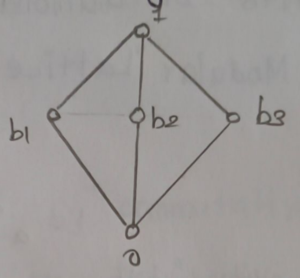
Hence the proof.

---

1) Check the Lattice given by the diagrams are distributive or not.



(a)



(b)

(i) To check the given lattice in (a) is distributive, we have to check the following distributive property.

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Consider, (a_1, a_2, a_3)

$LHS = a \vee (b \wedge c) = a_1 \vee (a_2 \wedge a_3) = a_1 \vee 0 = a_1$

$RHS = (a \vee b) \wedge (a \vee c) = (a_1 \vee a_2) \wedge (a_1 \vee a_3) = 1 \wedge a_3 = a_3$

$\therefore LHS \neq RHS$

$\therefore$ The Lattice given in (a) is not distributive.

(ii) Consider (b_1, b_2, b_3)

$LHS = a \vee (b \wedge c) = b_1 \vee (b_2 \wedge b_3) = b_1 \vee 0 = b_1$

$RHS = (a \vee b) \wedge (a \vee c) = (b_1 \vee b_2) \wedge (b_1 \vee b_3) = 1 \wedge 1 = 1$

$\therefore LHS \neq RHS$

$\therefore$ The Lattice given in (b) is not distributive.

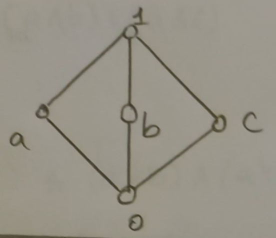
Modular Lattice:

A Lattice $(L, \wedge, \vee)$ is said to be Modular Lattice if it satisfies the following conditions. If $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c \ \forall \ a, b, c \in L$.

Examples:

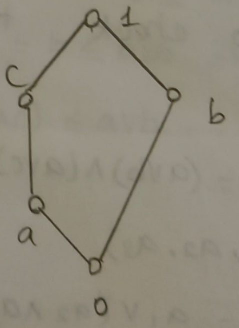
① $(P(A), \cup, \cap)$ is an example of both Modular Lattice and distributive Lattice.

② M_5 or Diamond Lattice is an example of Modular Lattice but non-distributive Lattice.



M_5

1) N_5 or Pentagon Lattice is an example of both non-modular and non-distributive Lattice.



Theorem:

Every distributive Lattice is modular, but not conve

Proof:

Let $(L, \wedge, \vee)$ be the given distributive Lattice.
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds good $\forall \ a, b, c \in L$.

If $a \leq c$ then $a \vee c = c$

$\therefore a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$

If $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$.

$\therefore$ Every distributive Lattice is modular.

But converse is not true.

that is Every modular Lattice need not be distributive.

For example, M_5 (Diamond) Lattice is modular but it is not distributive.

Theorem:

In any distributive Lattice $(L, \wedge, \vee)$ prove that
$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$.

Proof:

LHS $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= (a \vee b) \wedge (c \vee b) \wedge (c \vee a)$ ($\because$ by commutative law)

$= (a \vee b) \wedge (c \vee (b \wedge a))$ ($\because$ by distributive law)

$= ((a \vee b) \wedge c) \vee ((a \vee b) \wedge (b \wedge a))$ ($\because$ by distributive law)

$= ((a \vee b) \wedge c) \vee ((a \vee b) \wedge (a \wedge b))$ ($\because$ by commutative law)

$= (c \wedge (a \vee b)) \vee ((a \wedge b) \wedge (a \vee b))$ ($\because$ by commutative law)

$= (c \wedge a) \vee (c \wedge b) \vee ((a \wedge b) \wedge a) \vee ((a \wedge b) \wedge b)$ ($\because$ by distributive law)

$= (c \wedge a) \vee (c \wedge b) \vee (a \wedge b) \vee (a \wedge b)$ ($\because$ by Idempotent law)

$= (c \wedge a) \vee (c \wedge b) \vee (a \wedge b)$ ($\because$ by Idempotent law)

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ ($\because$ by commutative law)

$=$ RHS.

Hence the proof.

Theorem:

State and prove Demorgan's law of lattice.

Statement:

If $(L, \wedge, \vee, 0, 1)$ is a complemented Lattice, then

(1) $(a \wedge b)' = a' \vee b'$ (or) $\overline{(a \wedge b)} = \bar{a} \vee \bar{b}$

(2) $(a \vee b)' = a' \wedge b'$ (or) $\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$

Proof:

Claim: $(a \wedge b)' = a' \vee b'$

To prove the above, it is enough to prove that

(i) $(a \wedge b) \wedge (a' \vee b') = 0$

(ii) $(a \wedge b) \vee (a' \vee b') = 1$

(i) LHS $= (a \wedge b) \wedge (a' \vee b')$

$= ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b')$. ($\because$ by distributive)

$= (a \wedge a' \wedge b) \vee (a \wedge b \wedge b')$

$= (0 \wedge b) \vee (a \wedge 0)$ $\quad$ ($\because a \wedge a' = 0$ and $b \wedge b' = 0$)

$= 0 \vee 0$

$= 0$ $\quad$ ($\because$ by Idempotent)

$=$ RHS

$\therefore (a \wedge b) \wedge (a' \vee b') = 0$

(ii) LHS $= (a \wedge b) \vee (a' \vee b')$

$= (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b'))$ ($\because$ by distributive)

$= (a \vee a' \vee b') \wedge (a' \vee b \vee b')$

$= (1 \vee b') \wedge (a' \vee 1)$ $\quad$ ($\because a \vee a' = 1$)

$= 1 \wedge 1$

$= 1$ $\quad$ ($\because$ by Idempotent)

$=$ RHS

$\therefore (a \wedge b) \vee (a' \vee b') = 1$

From (i) and (ii), $(a \wedge b)' = a' \vee b'$

Claim: $(a \vee b)' = a' \wedge b'$

It is enough to prove that

(i) $(a \vee b) \wedge (a' \wedge b') = 0$

(ii) $(a \vee b) \vee (a' \wedge b') = 1$

(i) LHS $= (a \vee b) \wedge (a' \wedge b')$

$= (a \wedge (a' \wedge b')) \vee (b \wedge (a' \wedge b'))$ ($\because$ by distributive)

$= (a \wedge a' \wedge b') \vee (a' \wedge b \wedge b')$

$= (0 \wedge b') \vee (a' \wedge 0)$

$= 0 \vee 0$

$= 0$ ($\because$ by Idempotent)

$=$ RHS

$\therefore (a \vee b) \wedge (a' \wedge b') = 0$

(ii) LHS $= (a \vee b) \vee (a' \wedge b')$

$= ((a \vee b) \vee a') \wedge ((a \vee b) \vee b')$ ($\because$ by distributive)

$= (a \vee a' \vee b) \wedge (a \vee b \vee b')$

$= (1 \vee b) \wedge (a \vee 1)$

$= 1 \wedge 1$

$= 1$ ($\because$ by Idempotent)

$=$ RHS

$\therefore (a \vee b) \vee (a' \wedge b') = 1$

From (i) and (ii), we get

$(a \vee b)' = a' \wedge b'$

Hence the proof.

Bounded Lattice:

Let $(L, \wedge, \vee)$ be a given Lattice. If it has both '0' element and '1' element then it is said to be bounded Lattice. It is denoted by $(L, \wedge, \vee, 0, 1)$.

E.g:

$(P(A), \subseteq)$ is a Bounded Lattice.

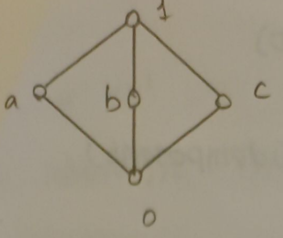
Here '0' element is ϕ, '1' element is A.

Complement:

Let $(L, \wedge, \vee, 0, 1)$ be a given bounded lattice. Let 'a' be any element of L. We say that 'b' is complement of a, if $a \wedge b = 0$ and $a \vee b = 1$ and 'b' is denoted by the symbol a'. (that is $b = a'$).

$\therefore a \wedge a' = 0, a \vee a' = 1$.

E.g:



Complement of a $= a'$ is b and c.

Complement of b $= b'$ is a and c

Complement of c $= c'$ is a and b.

Complemented Lattice:

A bounded Lattice $(L, \wedge, \vee, 0, 1)$ is said to be complemented Lattice if every element of L has atleast one complement.

Complete Lattice:

A Lattice $(L, \wedge, \vee)$ is said to be complete lattice if every non-empty subsets of L has both glb and lub.

E.g: $\{P(A), \subseteq\}$ is an example of Complete lattice

Theorem:

Prove that in a complemented distributive Lattice, complement is unique (or) If $(L, \wedge, \vee, 0, 1)$ is a distributive Lattice, then each element $x \in L$ has atmost one complement.

Proof:

Let us assume x and y are two complement of a.

Since x is a complement of a, we have

$a \wedge x = 0$, $a \vee x = 1$ —(1).

since y is a complement of a, we have

$a \wedge y = 0$, $a \vee y = 1$ —(2).

Now $x = x \vee 0$

$= x \vee (a \wedge y)$ ($\because$ by (2))

$= (x \vee a) \wedge (x \vee y)$ ($\because$ by distributive)

$= (a \vee x) \wedge (x \vee y)$ ($\because$ by commutative)

$= 1 \wedge (x \vee y)$ ($\because$ by (1))

$x = x \vee y$ — (3)

Now, $y = y \vee 0$

$= y \vee (a \wedge x)$ ($\because$ by (1))

$= (y \vee a) \wedge (y \vee x)$ ($\because$ by distributive)

$= (a \vee y) \wedge (y \vee x)$ ($\because$ by commutative)

$= 1 \wedge (y \vee x)$

$y = y \vee x$ — (4)

From (3) and (4), $x = x \vee y = y \vee x = y$.

$\Rightarrow x = y$

Hence complement is unique. Hence the proof.

---

1) In a complemented distributive Lattice show that the following are equivalent. $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$ (or) Prove that the above are equivalent.

(i) $\Rightarrow$ (ii)

Let $a \leq b$

Then $a \wedge b = a$ and $a \vee b = b$ — (1)

$a \wedge b' = (a \wedge b) \wedge b'$ ($\because$ by (1))

$= a \wedge b \wedge b'$

$= a \wedge 0$ ($\because b \wedge b' = 0$)

$= 0$

Hence $a \leq b \Rightarrow a \wedge b' = 0$.

(ii) $\Rightarrow$ (iii)

Let $a \wedge b' = 0$

Take complement on both sides we have

$(a \wedge b')' = 0'$

$a' \vee b = 1$

$\therefore a \wedge b' = 0 \Rightarrow a' \vee b = 1$

(iii) $\Rightarrow$ (iv)

Let $a' \vee b = 1$

$\Rightarrow (a' \vee b) \wedge b' = 1 \wedge b'$

$\Rightarrow (a' \wedge b') \vee (b \wedge b') = b'$ ($\because$ by distributive)

$\Rightarrow (a' \wedge b') \vee 0 = b'$ ($\because b \wedge b' = 0$)

$\Rightarrow a' \wedge b' = b'$

$\Rightarrow b' \leq a'$

Hence $a' \vee b = 1 \Rightarrow b' \leq a'$

(iv) $\Rightarrow$ (i)

Let $b' \leq a'$

Then $a' \wedge b' = b'$

Take complement on both sides, we have

$(a' \wedge b')' = (b')'$

$a \vee b = b$

$\Rightarrow a \leq b$

Hence $b' \leq a' \Rightarrow a \leq b$

Hence the proof.

Boolean Algebra

A complemented distributive Lattice is called Boolean Algebra. A boolean algebra is a distributive Lattice with '0' (GLB) element and '1' (LUB) element in which each element has a complement.

A Boolean algebra is a non-empty set with 2 binary operations $\wedge$ and $\vee$ is satisfied by the following conditions.

$\forall a, b, c \in L$.

① L_1 : $a \wedge a = a$, $a \vee a = a$

② L_2 : $a \wedge b = b \wedge a$, $a \vee b = b \vee a$

③ L_3 : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$

④ L_4 : $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

⑤ D_1 : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

⑥ D_2 : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

⑦ There exist elements '0' and '1' such that

$a \wedge 0 = a$, $a \vee 0 = a$, $a \wedge 1 = a$ and $a \vee 1 = 1$ $\forall a \in L$.

⑧ For all $a \in L$, there exist corresponding element $a \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 1$.

Note:

Boolean Algebra is denoted by $(B, \wedge, \vee, 0, 1)$.

Example:

$(P(A), \cup, \cap)$ is a Boolean Algebra, where A is any finite set.

Here '0' element is ϕ.

'1' element is A.

Complement of $a = a' = A - a$

Note:

In a Boolean Algebra

$0' = 1$

$1' = 0$